ON q-OPTIMAL SIGNED MARTINGALE MEASURES IN EXPONENTIAL LÉVY MODELS

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Abstract. We give a sufficient condition to identify the q-optimal signed martingale measures in exponential Lévy models. As a consequence we find that the q-optimal signed martingale measures can be equivalent only, if the tails for upward jumps are extraordinarily light. Moreover, we derive convergence of the q-optimal signed martingale measures to the entropy minimal martingale measure as q approaches one. Finally, some implications for portfolio optimization are discussed.

1. Introduction

Recently, the characterization of q-optimal equivalent martingale measures in market models with jumps has been studied in several papers. Jeanblanc *et al.* (2007) consider exponential Lévy processes. By a pointwise minimization procedure and formal application of the Kuhn-Tucker theorem they give sufficient conditions to identify the q-optimal equivalent martingale measures by finding a root of a deterministic equation, provided this root satisfies a positivity condition related to the Lévy measure. Their procedure generalizes and simplifies earlier work by Fujiwara and Miyahara (2003) and Esche and Schweizer (2005) for the entropy minimal martingale measure. As demonstrated by Choulli *et al.* (2007) the same type of root and positivity condition can be used to characterize minimal Hellinger martingale measures of order q in a general semimartingale framework. In a parallel development Kohlmann and Xiong (2007) derive a semimartingale backward equation with jumps to identify q-optimal equivalent martingale measures extending methods developed by Mania *et al.* (2002) for models with continuous trajectories.

It is well known, however, that in, the presence of jumps, q-optimal martingale measures may fail to be equivalent, but belong to the larger class of

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signed martingale measures. As all mentioned papers exploit representation results for densities of equivalent martingale measures as stochastic exponentials, these techniques cannot be generalized to study characterization results for q-optimal signed martingale measures. As a main result of this paper we demonstrate that, under suitable conditions on q, the positivity condition in Jeanblanc *et al.* (2007) can be dropped. The root condition is sufficient to identify the q-minimal signed martingale measure in exponential Lévy models. Our proof is based on a verification procedure in terms of a hedging problem, which holds true in a general semimartingale setting and is based on duality for convex optimization. In the one-dimensional case we give necessary and sufficient conditions for the existence of a root in terms of the Lévy measure. Roughly speaking, the existence of a root holds true if the tails for upward jumps are moderately sized, while the positivity assumption in Jeanblanc *et al.* (2007) typically requires the tails of the upward jumps to be extraordinarily light. Therefore, in many practically relevant models such as generalized hyperbolic models or the Merton model, the q-optimal martingale measure is typically a signed one.

As a second contribution we study the convergence of the q-optimal signed martingale measures to the entropy minimal equivalent martingale measure for one-dimensional exponential Lévy processes. This convergence problem was first treated by Grandits and Rheinländer (2002) in semimartingale models with continuous trajectories. Our results significantly generalize the corresponding results in Jeanblanc *et al.* (2007), even in situations where the q-optimal measures are equivalent. Theorem 2.6 below entails, for instance, convergence under the assumption of bounded upward jump heights without any assumptions on the activity of the jumps. When the positivity condition fails and the q-optimal measures are signed, Theorem 2.6 provides, to the best of our knowledge, the first convergence result to the minimal entropy measure. Finally, the convergence results for the measures are applied to establish convergence of an approximating sequence to the exponential utility maximization problem, complementing earlier work by Kohlmann and Niethammer (2007) and Niethammer (2007).

The paper is organized as follows: After an extended discussion of the results in Section 2, Sections 3 and 4 are devoted to the verification of the q-optimal signed martingale measure and the convergence to the entropy minimal one, respectively. Section 5 collects some consequences for the exponential utility maximization problem. Some proofs are postponed to an Appendix.

2. Discussion of the Results

We recall the standard setting for exponential Lévy models. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T \in (0, \infty)$ a finite time horizon, and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ a filtration satisfying the usual conditions, i.e. right-continuity and completeness. $\check{X}$ is supposed to be an $\mathbb{R}^n$-valued Lévy process with characteristic triplet $(\sigma\sigma^*, \nu, b)$ on $(\Omega, \mathcal{F}, P)$. By the Lévy-Itô-decomposition, (see e.g Cont and Tankov, 2004), $\check{X}$ has the following form:

$$\check{X}_t = bt + \sigma W_t + \int_0^t \int_{\|x\|>1} xN(dx, ds) + \int_0^t \int_{\|x\|\leq 1} x\tilde{N}(dx, ds) \tag{1}$$

where W is a n-dimensional Brownian motion, N is a Poisson random measure with intensity measure $\nu(dx)dt$, and $\tilde{N}(dx, dt) = N(dx, dt) - \nu(dx)dt$. Here, the Lévy measure ν is defined on $\mathbb{R}_0^n := \mathbb{R}^n \setminus \{0\}$.

We suppose that a discounted market with n assets is given by

$$S_t = \text{diag}(S_0^{(1)}, \ldots, S_0^{(n)})e^{\check{X}_t}, \quad t \in [0, T], \ S_0^{(i)} > 0. \tag{2}$$

The following standing assumptions are in force throughout the paper:

Assumption 2.1. *(i) The filtration $\mathbb{F}$ coincides with $\mathbb{F}^{\check{X}}$, the completion of the filtration generated by the Lévy process $\check{X}$.*

(ii) $E[|S(t)|] < \infty$

The second assumption guarantees that S is a special semimartingale with decomposition $S_t = S_0 + M_t + A_t$, where

$$dM_t = \mathbf{S}_{t-}(\sigma dW_t + \int_{\mathbb{R}_0^n} (e^x - \mathbf{1})\tilde{N}(dx, dt))$$

and

$$dA_t = \mathbf{S}_{t-}(-\beta + \int_{\mathbb{R}_0^n} (e^x - \mathbf{1} - x1_{\|x\|\leq 1})\nu(dx))dt.$$

Here, $\beta = -(b + \frac{1}{2}\sum_j \sigma_{\cdot,j}^2)$ and $\mathbf{S} = diag(S^{(1)}, ..., S^{(n)})$. $\mathbf{1}$ denotes the vector in $\mathbb{R}^n$ having all entries equal to one, and expressions such as e^x are to be interpreted componentwise, i.e. $e^x = (e^{x_1}, ..., e^{x_n})'$.

To properly explain the problem to be discussed in this paper, we first introduce the set of *q-integrable density processes of signed local martingale measures*,

$$\mathcal{D}_s^q = \{Z \in \mathcal{U}^q | E(Z_T) = 1, \ SZ \text{ is a local } P\text{-martingale}\},$$

where $\mathcal{U}^q$ denotes the set of $\mathbb{R}$-valued $L^q(\Omega, P)$-uniformly integrable martingales. Clearly $Z \in \mathcal{D}_s^q$ can be identified with a signed measure setting $dQ_Z = Z_T dP$. We will first study the following problem:

(Min$_{s,q}$) Find $Z^{(q)} \in \mathcal{D}_s^q$ such that

$$E[|Z_T^{(q)}|^q] = \inf_{Z\in\mathcal{D}_s^q} E[|Z_T|^q].$$

$dQ^{(S,q)} = Z_T^{(q)} dP$ is called the *q-optimal signed martingale measure* $Q^{(S,q)}$ (qSMM).

Recently, Jeanblanc *et al.* (2007) discussed the closely related problem, where minimization is considered over the subset of density processes corresponding to *q-integrable equivalent local martingale measures*

$$\mathcal{D}_e^q = \{Z \in \mathcal{D}_s^q \ Z_T > 0 \ P\text{-a.s.}\},$$

only. Hence,

(Min$_{e,q}$) Find $\tilde{Z}^{(q)} \in \mathcal{D}_e^q$ such that

$$E[|\tilde{Z}_T^{(q)}|^q] = \inf_{Z \in \mathcal{D}_e^q} E[|Z_T|^q].$$

$dQ^{(e,q)} = \tilde{Z}_T^{(q)} dP$ is called the *q-optimal equivalent martingale measure* $Q^{(e,q)}$ (qEMM).

Apart from characterizing the q-optimal signed martingale measures, we study their behaviour as q tends to 1. For models with continuous trajectories it was shown in Grandits and Rheinländer (2002) that, under some technical assumption, the q-optimal equivalent martingale measures converge to the entropy minimal martingale measure (as $q \to 1$). The latter one is defined to be the equivalent local martingale measure, which minimizes the entropy relative to P over the set of all equivalent local martingale measures with finite relative entropy:

(Min$_{e,\log}$) Find $Z^{\min} \in \mathcal{D}_e^{\log}$ such that

$$E[Z_T^{\min} \log Z_T^{\min}] = \inf_{Z \in \mathcal{D}_e^{\log}} E[Z_T \log Z_T].$$

where correspondingly

$$\mathcal{D}_e^{\log} = \{Z \in \mathcal{D}_e^1, \ E(Z_T \log Z_T) < \infty\}.$$

$dQ_{\min} = Z_T^{\min} dP$ is called the *minimal entropy martingale measure* $Q_{\min}$ (MEMM).

Jeanblanc *et al.* (2007) discussed, for the first time, the convergence of the qEMMs to the MEMM in the framework of exponential Lévy models under some technical conditions. To this end, they first propose the following sufficient condition to identify the qEMM, which we here split into two parts for later reference:

Assumption 2.2. *(C_q)*
$\mathbf{C_q^-}$: *There exists an $\theta_q \in \mathbb{R}^n$ such that*

$$\mathrm{eg}_q(x) := ((q-1)\theta_q'(e^x - \mathbf{1}) + 1)^{\frac{1}{q-1}}$$

defines a real-valued function on the support on ν which satisfies

$$\sigma\sigma'\theta_q + \int_{\mathbb{R}_0^n} (e^x - \mathbf{1})\mathrm{eg}_q(x) - x1_{\|x\| \le 1}\nu(dx) = \beta \tag{3}$$

and

$$\int_{\mathbb{R}_0^n} |(\mathrm{eg}_q^q(x)) - 1 - q(\mathrm{eg}_q(x) - 1)|\nu(dx) < \infty. \tag{4}$$

$\mathbf{C_q^+}$: $(q-1)\theta_q'(e^y - \mathbf{1}) + 1 > 0, \nu$-a.s.

If C_q^- and C_q^+ are satified, we say that C_q holds.

Theorem 2.1 (Jeanblanc *et al.* (2007), Theorem 2.9)**.** *Suppose C_q holds. Then the qEMM exists and is given by*

$$\mathcal{E}(\theta_q'\sigma, \mathrm{eg}_q - 1),$$

where $\mathcal{E}(f,g)$ denotes the stochastic exponential with Girsanov parameters f, g, i.e.

$$\begin{aligned}\mathcal{E}_t(f,g) &= \exp\{\int_0^t f(s)dW_s - \frac{1}{2}\int_0^t \|f(s)\|^2 ds + \int_0^t \int_{\mathbb{R}_0^n} g(s,x)\tilde{N}(dx,ds)\} \\ &\quad \times \prod_{s \le t}(1 + g(s, \Delta\check{X}(s)))e^{-g(s,\Delta\check{X}(s))}.\end{aligned}$$

In the one-dimensional case, we now give some necessary conditions for C_q. In particular, we find that C_q only holds, if the tails for upward jumps are extraordinarily light or the MEMM does not exist.

Proposition 2.2. *Suppose $n = 1$ and P is not a martingale measure. Then:*
(i) If C_q holds for some $q > 1$, then

$$\int_{x \ge 1} e^{\theta e^x}\nu(dx) < \infty \tag{5}$$

for some $\theta > 0$ or the MEMM does not exist.
(ii) If C_q holds for some $q > 1$, then

$$\int_{\mathbb{R}_0} (e^x - 1) - x 1_{|x| \le 1}\nu(dx) + (b + \frac{1}{2}\sigma^2) < 0 \tag{6}$$

or upward jumps are bounded, i.e. $\nu([L, \infty)) = 0$ for some $L > 0$.

Remark 2.1. *Suppose $n = 1$.*
(i) Note that most of the concrete models discussed in the literature, such as generalized hyperbolic models or the popular jump-diffusion models by Merton or Kou satisfy

$$\int_{x \ge 1} e^{\theta e^x}\nu(dx) = \infty$$

for all $\theta > 0$. Hence, C_q and the existence of the MEMM cannot hold simultaneously for these models.

(ii) In condition (6) upward jumps are exponentially weighted and downward jumps are exponentially damped. Hence,

$$\int (e^x - 1) - x 1_{|x| \le 1}\nu(dx)$$

can become negative only, if the Lévy measure gives much more weight to negative jumps than to positive jumps, leading to an extreme gain-loss asymmetry in the jumps. In such situation we expect that the deterministic trend b is large to compensate for the risk of downward jumps. So condition (6) may be rather unlikely to occur.

An examination of the *proof*, which is postponed to the appendix, shows, that the positivity assumption C_q^+ is particularly restrictive. Indeed, C_q^+ rules out negative values for θ_q, if the upward jumps are unbounded. So let us now assume that $q = \frac{2m}{2m-1}$ for some $m \in \mathbb{N}$. Then $(q-1) = 2m-1$ and hence the $(q-1)$th root in the definition of C_q^- defines a real valued function even when $(q-1)\theta_q'(e^y - \mathbf{1}) + 1$ is negative. Hence, in this situation $\mathcal{E}(\theta_q'\sigma, \mathrm{eg}_q - 1)$ is a well-defined real-valued martingale, exploiting the fact that the stochastic exponential of a Lévy martingale is always a martingale (see Cont and Tankov, 2004, Prop. 8.23). However, it defines a signed measure only and not an equivalent one, when C_q^+ fails. It is a good guess that $\mathcal{E}(\theta_q'\sigma, \mathrm{eg}_q - 1)$ is then the q-optimal signed martingale measure:

Theorem 2.3. *Suppose that $q = \frac{2m}{2m-1}$ for some $m \in \mathbb{N}$ and that C_q^- holds. Then,*

$$Z^{(q)} = \mathcal{E}(\theta_q'\sigma, \mathrm{eg}_q - 1)$$

is the density process of qSMM.

The proof will be given in Section 3.

Remark 2.2. *The proof of Theorem 2.9 in Jeanblanc et al. (2007) is very intuitive: Exploiting that all equivalent martingale measures can be represented by stochastic exponentials, they first show that passing from random to appropriate deterministic and time-independent Girsanov parameters, the L^q-norm can always be reduced. Then C_q is derived by a formal application of the Kuhn-Tucker theorem to the problem of finding optimal deterministic and time-independent Girsanov parameters. This line of arguments cannot be applied to the problem of finding the q-optimal signed martingale measure, as signed measures cannot be represented by stochastic exponentials in general. Instead we will prove Theorem 2.3 by a verification procedure that makes use of a hedging argument and duality.*

The following proposition gives a complete characterization of condition C_q^- in the one-dimensional case. Roughly speaking, C_q^- holds, if and only if the tails of upward jumps are moderately sized, and hence is much weaker than C_q. Its proof will also be postponed to the appendix.

Proposition 2.4. *Suppose $n = 1$, $q(m) = \frac{2m}{2m-1}$, P is not a martingale measure, and the set of equivalent martingale measures is nonempty. Then, $C_{q(m)}^-$ holds for $m \in \mathbb{N}$, if and only if*

$$\int_{x \geq 1} e^{2mx} \nu(dx) < \infty. \tag{7}$$

Example 2.1. *Suppose* $n = 1$.
(i) If $\nu(dx)$ *behaves (up to a slowly varying function) as* $e^{-\lambda_+ x}dx$ *for* $x \to \infty$, *then* $C^-_{q(m)}$ *holds for* $m < \lambda_+/2$ *and fails for* $m > \lambda_+/2$. *However,* C_q *fails for all* q, *if*

$$\int_{\mathbb{R}_0} (e^x - 1) - x1_{|x|\leq 1}\nu(dx) + (b + \frac{1}{2}\sigma^2) > 0.$$

This tail behavior is inherent in generalized hyperbolic models and the Kou model.
(ii) If there are constants $\eta_0, \eta_1 > 0$ *such that*

$$\int_{x\geq 1} e^{\eta_0 x^{1+\eta_1}}\nu(dx) < \infty, \tag{8}$$

then $C^-_{q(m)}$ *holds for all* $m \in \mathbb{N}$. *However* C_q *fails for all* q, *if the upward jumps are not bounded and*

$$\int_{\mathbb{R}_0} (e^x - 1) - x1_{|x|\leq 1}\nu(dx) + (b + \frac{1}{2}\sigma^2) > 0.$$

A popular model, which satisfies (8) and has unbounded upward jumps is the Merton model.

Our study on the convergence of the q-optimal signed martingale measures to the minimal entropy martingale measure for exponential Lévy models will make use of the following characterization of the minimal entropy measure based on Condition C:

Assumption 2.3. *(C)*
There exists a vector $\theta_e \in \mathbb{R}^n$ *satisfying*

$$\int_{\mathbb{R}^n_0} \|(e^x - \mathbf{1})e^{\theta'_e(e^x-\mathbf{1})} - x1_{\|x\|\leq 1}\|\nu(dx) < \infty \tag{9}$$

and

$$\theta'_e\sigma\sigma' + \int_{\mathbb{R}^n_0} (e^x - \mathbf{1})e^{\theta'_e(e^x-\mathbf{1})} - x1_{\|x\|\leq 1}\nu(dx) = \beta. \tag{10}$$

Theorem 2.5. *(i) If condition* C *is satisfied, then the entropy minimal martingale measure is given by*

$$\mathcal{E}(\theta'_e\sigma, e^{\theta_e(e^x-1)} - 1).$$

(ii) If $n = 1$ *and there is no* θ_e *satisfying* C, *then the entropy minimal martingale measure does not exist.*

Item (i) is due to Fujiwara and Miyahara (2003) for $n = 1$ and Esche and Schweizer (2005) for the multidimensional case. Item (ii) was proved by Hubalek and Sgarra (2006).

In the one-dimensional case, we derive the following result concerning the convergence of the q-optimal martingale measures for exponential Lévy models, which will be proved in Section 4:

Theorem 2.6. *Suppose $n = 1$, the minimal entropy martingale measure exists, and there is a $\delta > 0$ such that θ_e, specified by condition C, satisfies*

$$\int_{x\geq 1} e^{(\max\{\theta_e, -0.28\,\theta_e\}+\delta)e^x}\nu(dx) < \infty. \tag{11}$$

Then:

(i) If $\theta_e > 0$ or upwards jumps are bounded, then C_q is satisfied for sufficiently small $q > 1$ and the q-optimal equivalent martingale measures converge to the minimal entropy martingale measure in $L^r(P)$, for some $r > 1$, as $q \downarrow 1$ (in the sense that the densities converge).

(ii) Suppose $q(m) = \frac{2m}{2m-1}$. If $\theta_e < 0$, then $C^-_{q(m)}$ is satisfied for all $m \in \mathbb{N}$ and the $q(m)$-optimal signed martingale measures converge to the minimal entropy martingale measure in $L^r(P)$, for some $r > 1$, as $m \uparrow \infty$.

Remark 2.3. *Suppose $n = 1$.*
(i) $\theta_e = 0$, if and only if P itself is a local martingale measure. In this case P obviously is q-optimal as well, and therefore convergence is trivial.

(ii) Item (i) of Theorem 2.6 significantly generalizes a result in Jeanblanc et al. (2007), which requires that jumps are of finite activity and upward jumps are bounded.

(iii) If in item (ii) of Theorem 2.6 the upward jumps are not bounded, then the $q(m)$-optimal signed martingale measures are not equivalent to P and nonetheless convergence holds.

(iv) The constant 0.28 in condition (11) can be replaced by any constant $a > 0$ such that $y \leq e^{ay} + 1$ for all $y \in \mathbb{R}$.

3. Verification of the q-Optimal Solution

This section is devoted to the proof of Theorem 2.3. As a first step we derive a verification theorem for the q-optimal signed martingale measure. This theorem does not rely on the Lévy setting, but holds for general semimartingale models.

Theorem 3.1. *Suppose $\hat{Z} \in \mathcal{D}^q_S$, $q = \frac{2m}{2m-1}$ and, for some $\tilde{x} < 2m$, the contingent claim*

$$X^{(2m)}(\hat{Z}) := 2m - 2m\hat{Z}_T^{\frac{1}{2m-1}}\left(\frac{2m-\tilde{x}}{2mE(\hat{Z}_T^{\frac{2m}{2m-1}})}\right) \tag{12}$$

is replicable with a predictable strategy ϑ (the number of shares held), such that

$$\|\vartheta\|_{L^{2m}(M)} := \|(\int_0^T \vartheta d[M]_t\vartheta')^{\frac{1}{2}}\|_{L^{2m}(\Omega,P)} < \infty, \tag{13}$$

$$\|\vartheta\|_{L^{2m}(A)} := \|\int_0^T |\vartheta dA_t|\|_{L^{2m}(\Omega,P)} < \infty. \tag{14}$$

Then $\hat{Z}$ is the density process of the q-optimal signed martingale measure.

From now on, $\mathcal{A}^{(2m)}$ denotes the set of predictable process ϑ such that (13)–(14) holds.

Remark 3.1. *Suppose $q = \frac{2m}{2m-1}$, $\vartheta \in \mathcal{A}^{(2m)}$, $x \in \mathbb{R}$, and consider the wealth process*

$$V_t(\vartheta, x) = x + \int_0^t \vartheta_u dS_u.$$

Then, by the Burkholder-Davis-Gundy inequality, $Z_t V_t(\vartheta, X)$ is a martingale for every $Z \in \mathcal{D}_S^q$. In particular,

$$E[Z_T V_T(\vartheta, x)] = x.$$

Therefore, a direct calculation shows, that the initial capital required to replicate $X^{(2m)}(\hat{Z})$ in (12) with a strategy of class $\mathcal{A}^{(2m)}$ is $\tilde{x}$ (provided replication is possible).

Proof of Theorem 3.1. Let $q = 2m/(2m-1)$. We consider the following maximization problems with utility function $u_{2m}(x) = -(1 - \frac{x}{2m})^{2m}$:

$$\begin{aligned}
\text{Max}_1:\ X^{(1)} &:= \arg\max\{E(u_{2m}(X));\ X \text{ s.t. } E(\hat{Z}_T X) \le \tilde{x}\} \\
\text{Max}_2:\ X^{(2)} &:= \arg\max\{E(u_{2m}(X));\ X \text{ s.t. } \forall Z \in \mathcal{D}_S^q:\ E(Z_T X) \le \tilde{x}\} \\
\text{Max}_3:\ X^{(3)} &:= \arg\max\{E(u_{2m}(X));\ X \in \Theta^{(2m),\tilde{x}}\}
\end{aligned}$$

where

$$\Theta^{(2m),\tilde{x}} = \left\{ X \in L^{2m}(\Omega, \mathcal{F}_T, P):\ \exists \vartheta \in \mathcal{A}^{(2m)} \text{ s.t. } X = \tilde{x} + \int_0^T \vartheta_u dS_u \right\}.$$

From Remark 3.1 we derive that

$$E(u_{2m}(X^{(1)})) \ge E(u_{2m}(X^{(2)})) \ge E(u_{2m}(X^{(3)})) \ge E(u_{2m}(X^{(2m)}(\hat{Z}))). \tag{15}$$

A straightforward calculation shows that the convex dual of u_{2m} is given by

$$\breve{u}_{2m}(y) = (2m-1)y^{2m/(2m-1)} - 2my. \tag{16}$$

Therefore standard duality theory can be applied to verify that $X^{(2m)}(\hat{Z})$ is the maximizer of problem Max_1, see Kohlmann and Niethammer (2007) for details. In particular, we observe that all inequalities turn into identities in formula (15).

In a next step we exploit that the problem Max_2 is dominated by its dual minimization problem (see e.g. Luenberger, 1969, p. 225), and thus, by (16),

$$E(u_{2m}(X(\hat{Z}))) = E(u_{2m}(X^{(2)})) \le \inf_{Z \in \mathcal{D}_S^q,\, y \ge 0} (E(\breve{u}_{2m}(y \cdot Z_T)) + \tilde{x}y)$$

$$= \inf_{y \ge 0} \left((2m-1)y^{2m/(2m-1)} \left(\inf_{Z \in \mathcal{D}_S^q} E[Z_T^{2m/(2m-1)}] \right) - (2m - \tilde{x})y \right) \tag{17}$$

Define

$$\hat{y}_{2m} = \left((2m - \tilde{x}) \left(2mE \left(\hat{Z}_T^{\frac{2m}{2m-1}} \right) \right)^{-1} \right)^{2m-1}.$$

As

$$X^{(2m)}(\hat{Z}) = 2m - 2m(\hat{y}_{2m}\hat{Z}_T)^{\frac{1}{2m-1}} = (u'_{2m})^{-1}(\hat{y}_{2m}\hat{Z}_T)$$

and

$$\check{u}_{2m}(y) = u_{2m}((u'_{2m})^{-1}(y)) - (u'_{2m})^{-1}(y)y,$$

we have, due to Remark 3.1,

$$E(u_{2m}(X(\hat{Z}))) = E(\check{u}_{2m}(\hat{y}_{2m} \cdot \hat{Z}_T)) + \tilde{x}\hat{y}_{2m}.$$

Consequently, the pair $(\hat{Z}, \hat{y}_{2m})$ is the minimizer of the problem on the right hand side of (17). This immediately implies that $\hat{Z}$ is the density process of the $2m/(2m-1)$-optimal signed martingale measure. □

Suppose now that $q = 2m/(2m-1)$ for some fixed $m \in \mathbb{N}$ and that C_q^- holds. Define

$$\hat{Z} = \mathcal{E}(\theta'_q\sigma, \mathrm{eg}_q - 1).$$

Then, by (4), $\hat{Z}$ is q-integrable. Moreover, applying the Doleans-Dade formula and integration by parts, one can easily verify that $\hat{Z}S$ is a local martingale thanks to (3). For this reason, (3) is referred to as the martingale condition, compare Kunita (2004). Hence, $\hat{Z} \in \mathcal{D}_S^q$.

Therefore, in order to prove Theorem 2.3, it remains to show that $X^{(2m)}(\hat{Z})$ is replicable within the class $\mathcal{A}^{(2m)}$ for $q = 2m/(2m-1)$ and some $\tilde{x} < 2m$. We will now derive a replicating strategy constructively. To this end we fix some $\tilde{x} < 2m$.

If $X^{(2m)}(\hat{Z})$ is replicable, then its conditional expectation under every martingale measure is equal to the price process of $X^{(2m)}(\hat{Z})$. So the candidate for the price process is

$$p_t := \hat{Z}_t^{-1} E[\hat{Z}_T X^{(2m)}(\hat{Z}) | \mathcal{F}_t].$$

Observe that by the definition of $\hat{Z}$ and $X^{(2m)}(\hat{Z})$

$$\hat{Z}_T X^{(2m)}(\hat{Z}) = 2m\mathcal{E}_T(\theta'_q\sigma, \mathrm{eg}_q - 1) - 2m\mathcal{E}_T^q(\theta'_q\sigma, \mathrm{eg}_q - 1) \left(\frac{2m - \tilde{x}}{2mE(\hat{Z}_T^q)} \right).$$

As the Girsanov parameters are deterministic, a simple computation yields,

$$\mathcal{E}_T^q(\theta'_q\sigma, \mathrm{eg}_q - 1) = \mathcal{E}_T(q\theta'_q\sigma, \mathrm{eg}_q^q - 1)E[\mathcal{E}_T^q(\theta'_q\sigma, \mathrm{eg}_q - 1)].$$

Consequently,

$$\begin{aligned} p_t &= 2m-(2m-\tilde{x})\mathcal{E}_t^{-1}(\theta_q'\sigma, \mathrm{eg}_q-1)\mathcal{E}_t(q\theta_q'\sigma, \mathrm{eg}_q^q-1) \\ &= 2m-(2m-\tilde{x})\mathcal{E}_t((q-1)\theta_q'\sigma, \mathrm{eg}_q^{q-1}-1) \\ &\quad \times \exp\left\{t\left(\int_{\mathbb{R}_0^n}[\mathrm{eg}_q(x)-\mathrm{eg}_q^q(x)+\mathrm{eg}_q^{q-1}(x)-1]\nu(dx)-(q-1)\|\theta_q'\sigma\|^2\right)\right\} \\ &=: 2m-(2m-\tilde{x})\hat{M}_t\hat{A}_t. \end{aligned}$$

Taking the definition of eg_q in condition C_q^- and the martingale condition (3) into account, we obtain,

$$\begin{aligned} \hat{A}_t &= \exp\left\{t(q-1)\theta_q'\left(-\beta+\int_{\mathbb{R}_0^n}(e^x-\mathbf{1}-x1_{\|x\|\le 1})\nu(dx)\right)\right\} \\ &= \exp\left\{(q-1)\theta_q'\int_0^t \mathbf{S}_{u-}^{-1}dA_u\right\} \end{aligned}$$

and

$$\hat{M}_t = \mathcal{E}_t((q-1)\theta_q'\sigma, (q-1)\theta_q'(e^{\cdot}-\mathbf{1})).$$

Hence, applying the Doleans-Dade formula to $\hat{M}$ and integration by parts, we get,

$$\begin{aligned} \hat{M}_t\hat{A}_t &= 1+(q-1)\theta_q'\int_0^t \mathbf{S}_{u-}^{-1}\hat{M}_{u-}\hat{A}_u dA_u \\ &\quad +(q-1)\theta_q'\left(\int_0^t \hat{M}_{s-}\hat{A}_s\sigma dW_s+\int_0^t\int_{\mathbb{R}_0^n}\hat{M}_{s-}\hat{A}_s(e^x-\mathbf{1})\tilde{N}(dx,ds)\right) \\ &= 1+(q-1)\theta_q'\int_0^t \mathbf{S}_{u-}^{-1}\hat{M}_{u-}\hat{A}_u dS_u. \end{aligned}$$

Consequently,

$$p_t = \tilde{x}+\int_0^t (2m-\tilde{x})(q-1)\theta_q'\hat{M}_{u-}\hat{A}_u\mathbf{S}_{u-}^{-1}dS_u.$$

As, by construction,

$$p_T = X^{(2m)}(\mathcal{E}(\theta_q'\sigma, \mathrm{eg}_q-1))$$

we have proved the following lemma.

Lemma 3.2. *Suppose that* $q=\frac{2m}{2m-1}$ *for some* $m\in\mathbb{N}$ *and that* C_q^- *holds. Define*

$$\begin{aligned} \vartheta_t^{(2m)} &= -\frac{2m-\tilde{x}}{2m-1}\mathcal{E}_t((q-1)\theta_q'\sigma, (q-1)\theta_q'(e^{\cdot}-\mathbf{1})) \\ &\quad \times \exp\left\{t(q-1)\theta_q'\left(-\beta+\int_{\mathbb{R}_0^n}(e^x-\mathbf{1}-x1_{\|x\|\le 1})\nu(dx)\right)\right\}\theta_q'\mathbf{S}_{t-}^{-1}. \end{aligned}$$

Then for $\tilde{x} \leq 2m$ *and* $\hat{Z} = \mathcal{E}(\theta_q'\sigma, \mathrm{eg}_q - 1)$ *the contingent claim*

$$X^{(2m)}(\hat{Z}) := 2m - 2mZ_q^{\frac{1}{2m-1}} \left(\frac{2m - \tilde{x}}{2mE(\hat{Z}_T^{\frac{2m}{2m-1}})} \right)$$

is replicable with initial wealth $\tilde{x}$ *and the predictable strategy* $\vartheta^{(2m)}$.

The proof of Theorem 2.3 will now be completed with the following lemma.

Lemma 3.3. *Suppose that* $q = \frac{2m}{2m-1}$ *for some* $m \in \mathbb{N}$ *and that* C_q^- *holds. Then the hedge* $\vartheta^{(2m)}$ *constructed in Lemma 3.2 belongs to the class* $\mathcal{A}^{(2m)}$.

Proof. Recall that by the definitions of eg_q and q

$$\mathcal{E}_t^{2m}((q-1)\theta_q'\sigma, (q-1)\theta_q'(e^{\cdot} - \mathbf{1})) = \mathcal{E}_t^{q/(q-1)}((q-1)\theta_q'\sigma, \mathrm{eg}_q^{q-1} - 1).$$

Hence, $\mathcal{E}_t((q-1)\theta_q'\sigma, (q-1)\theta_q'(e^{\cdot} - \mathbf{1}))$ is a $(2m)$-integrable martingale under C_q^- thanks to (4). By Doob's inequality, we immediately observe that $\vartheta^{(2m)}$ satisfies (14). To prove (13), we first notice that, for some constant K_q,

$$\begin{aligned} & E\left[\left(\int_0^T \vartheta^{(2m)} d[M]_t (\vartheta^{(2m)})' \right)^{\frac{2m}{2}} \right] \\ \leq\ & K_q E\Bigg[\Big[\int_0^{\cdot} \mathcal{E}_t((q-1)\theta_q'\sigma, (q-1)\theta_q'(e^{\cdot} - \mathbf{1})) \\ & \times d\Big((q-1)\theta_q'\sigma W_t + (q-1)\theta_q' \int_0^t \int_{\mathbb{R}_0^n} (e^x - \mathbf{1})\tilde{N}(dx, ds) \Big) \Big]_T^{\frac{2m}{2}} \Bigg] \\ =\ & K_q E\left[\left[\mathcal{E}((q-1)\theta_q'\sigma, (q-1)\theta_q'(e^{\cdot} - \mathbf{1})) - 1 \right]_T^{\frac{2m}{2}} \right], \end{aligned}$$

applying the Doleans-Dade formula for the last identity. By the Burkholder-Davis-Gundy inequality and since $\mathcal{E}((q-1)\theta_q'\sigma, (q-1)\theta_q'(e^{\cdot} - \mathbf{1}))$ is $(2m)$-integrable, the expression on the right hand side is finite. □

4. Convergence to the Minimal Entropy Measure

In this section we will prove Theorem 2.6, hence convergence of the q-optimal signed martingale measures to the entropy minimal martingale measure. Throughout the section we shall assume that the exponential Lévy process is one-dimensional. The proof will proceed through three steps: (i) Convergence of the roots θ_q to θ_e; (ii) convergence of the q-optimal densities in probability; (iii) convergence of the q-optimal densities in $L^r(\Omega, P)$ for some $r > 1$.

However, we will first study some properties of the following mappings which were introduced in Jeanblanc *et al.* (2007) (in the notation of Niethammer, 2007):

$$\Phi(q,\theta) = -\beta + \sigma^2\theta + \int_{\mathbb{R}} (e^x - 1)((q-1)\theta(e^x - 1) + 1)^{\frac{1}{q-1}} - x1_{|x|\leq 1}\nu(dx)$$

which are, for fixed $q > 1$, defined on the domain

$$\begin{aligned}\mathrm{Dom}_q \quad := \quad & \{\theta;\ ((q-1)\theta(e^x - 1) + 1)^{\frac{1}{q-1}} \in \mathbb{R}, \\ & \int_{x\geq 1} |(e^x - 1)((q-1)\theta(e^x - 1) + 1)^{\frac{1}{q-1}}|\nu(dx) < \infty\}\end{aligned}$$

and

$$\Phi_e(\theta) = \Phi(e,\theta) = -\beta + \sigma^2\theta + \int_{\mathbb{R}} (e^x - 1)e^{\theta(e^x-1)} - x1_{|x|\leq 1}\nu(dx)$$

defined on

$$\mathrm{Dom}_e := \{\theta;\ \int_{x\geq 1} (e^x - 1)e^{\theta(e^x-1)}\nu(dx) < \infty\}$$

Lemma 4.1. *(i) Suppose* $\sigma \neq 0$ *or* $\nu(\mathbb{R}\backslash\{0\}) \neq 0$. *Then the mappings* $\Phi(q,\cdot)$ *and* Φ_e *are continuous and strictly increasing on their respective domains.*
(ii) For all $q > 1$, $0 \in \mathrm{Dom}_q \cap \mathrm{Dom}_e$ *and* $\Phi(q,0) = \Phi_e(0)$.
(iii) Suppose $q(m) = \frac{2m}{2m-1}$ *and*

$$\int_{x\geq 1} e^{2mx}\nu(dx) < \infty.$$

Then, $\mathrm{Dom}_{q(m)} = \mathbb{R}$.

Proof. (i) follows directly from the monotone convergence theorem.
(ii) is trivial.
(iii) We abbreviate $q = q(m)$. Since $\frac{1}{q-1} = 2m - 1$, there is a constant C_m such that for $x > 1$ and $\theta \in \mathbb{R}$

$$|(e^x - 1)((q-1)\theta(e^x - 1) + 1)^{\frac{1}{q-1}}| \leq \frac{C_m}{2m-1}|\theta|(e^x - 1)^{2m} + C_m(e^x - 1),$$

which is ν-integrable over $(1,\infty)$ by assumption. □

Note that the existence of an equivalent martingale measure implies that the assumption of item (i) above is in force.

Lemma 4.2. *Suppose* $\theta \in \mathrm{Dom}_e$.
(i) If $\theta > 0$ *or if the upward jumps are bounded, then* $\theta \in \mathrm{Dom}_q$ *for sufficiently small* $q > 1$ *and*

$$\lim_{q\downarrow 1} \Phi(q,\theta) = \Phi_e(\theta).$$

(ii) Suppose $\theta < 0$ *and*

$$\int_{x\geq 1} e^{0.28\,|\theta|e^x}\nu(dx) < \infty.$$

Then $\theta \in \mathrm{Dom}_{q(m)}$ *for all* $m \in \mathbb{N}$ *and*

$$\lim_{m\uparrow\infty} \Phi(q(m),\theta) = \Phi_e(\theta).$$

Proof. (i) Suppose first that $\theta > 0$. Then, there is a q_0 such that for all $1 < q < q_0$, $(q-1)\theta(e^x-1)+1 > 0$ for all $x \in \mathbb{R}$. This immediately implies $\theta \in \mathrm{Dom}_q$ for all $1 < q < q_0$, since θ belongs to Dom_e. Notice that $((q-1)\theta(e^x-1)+1)^{\frac{1}{q-1}}$ is monotonically decreasing in $q < q_0$. Hence , by monotone convergence, we get as $q \downarrow 1$,

$$\begin{aligned}
&\int_{x>0} (e^x-1)((q-1)\theta(e^x-1)+1)^{\frac{1}{q-1}} - x1_{|x|\le 1}\nu(dx) \\
\to &\int_{x>0} (e^x-1)e^{\theta(e^x-1)} - x1_{|x|\le 1}\nu(dx), \\
&\int_{x<0} (e^x-1)((q-1)\theta(e^x-1)+1)^{\frac{1}{q-1}} - x1_{|x|\le 1}\nu(dx) \\
\to &\int_{x<0} (e^x-1)e^{\theta(e^x-1)} - x1_{|x|\le 1}\nu(dx).
\end{aligned}$$

Since $\theta \in \mathrm{Dom}_e$, both limits on the right hand side are finite and, hence, can be combined to get

$$\lim_{q\downarrow 1} \Phi(q,\theta) = \Phi_e(\theta).$$

Now suppose $\theta < 0$ and some $K > 0$ is fixed. Then, for sufficiently small q, $(q-1)\theta(e^x-1)+1 > 0$ for all $x \le K$. Hence, we obtain, analogously to the case $\theta > 0$,

$$\begin{aligned}
&\int_{\{x\in\mathbb{R}_0;\, x\le K\}} (e^x-1)((q-1)\theta(e^x-1)+1)^{\frac{1}{q-1}} - x1_{|x|\le 1}\nu(dx) \\
\to &\int_{\{x\in\mathbb{R}_0;\, x\le K\}} (e^x-1)e^{\theta(e^x-1)} - x1_{|x|\le 1}\nu(dx) \qquad (18)
\end{aligned}$$

as $q \downarrow 1$. Hence,

$$\lim_{q\downarrow 1} \Phi(q,\theta) = \Phi_e(\theta),$$

if the upward jumps are bounded by K.

(ii) Let $\theta < 0$. By Lemma 4.1, (iii), we have $\theta \in \mathrm{Dom}_{q(m)}$ for all $m \in \mathbb{N}$. In view of (18) it remains to show that

$$\int_{x>1} (e^x-1)((q(m)-1)\theta(e^x-1)+1)^{\frac{1}{q(m)-1}}\nu(dx) \to \int_{x>1} (e^x-1)e^{\theta(e^x-1)}\nu(dx)$$

as m tends to infinity. This follows from the dominated convergence theorem, since on the set $\{x \ge 1; (q(m)-1)\theta(e^x-1) \ge -2\}$

$$|((q(m)-1)\theta(e^x-1)+1)^{\frac{1}{q(m)-1}}| \le 1$$

and on the set $\{x \geq 1; (q(m)-1)\theta(e^x-1) < -2\}$

$$\begin{aligned} & |((q(m)-1)\theta(e^x-1)+1)^{\frac{1}{q(m)-1}}| \\ \leq \;& ((q(m)-1)|\theta|(e^x-1)-1)^{\frac{1}{q(m)-1}} \leq e^{0.279\,|\theta|(e^x-1)}, \end{aligned}$$

since $y \leq e^{0.279\,y}+1$ for all $y \in \mathbb{R}$. □

Remark 4.1. *Suppose $\theta < 0$ and define the sequence x_m by*

$$(q(m)-1)|\theta|(e^{x_m}-1) = e+1.$$

Then, the sequence x_m is unbounded and

$$|((q(m)-1)\theta(e^{x_m}-1)+1)^{\frac{1}{q(m)-1}}| = e^{(e+1)^{-1}|\theta|(e^{x_m}-1)}$$

for all $m \in \mathbb{N}$. Hence, the estimates used to justify dominated convergence in the above proof are rather tight.

With this lemma at hand, we can prove the convergence of the Girsanov parameters θ_q to θ_e:

Theorem 4.3. *(i) Under the assumptions of Theorem 2.6, (i), C_q is satisfied for sufficiently small $q > 1$ and $\theta_q \to \theta_e$ holds as $q \downarrow 1$.*
(ii) Under the assumptions of Theorem 2.6, (ii), $C^-_{q(m)}$ is satisfied for all $m \in \mathbb{N}$ and $\theta_{q(m)} \to \theta_e$ holds as $m \uparrow \infty$.

Proof. (i) Due to assumption (11), we have $\theta_e - \epsilon, \theta_e + \epsilon \in \mathrm{Dom}_e$ for all $\epsilon < \delta$. By the strict monotonicity of Φ_e on its domain we derive

$$\Phi_e(\theta_e - \epsilon) < 0 = \Phi(\theta_e) < \Phi_e(\theta_e + \epsilon).$$

Choosing ϵ sufficiently small we can guarantee (due to positivity of θ_e, resp. boundedness of the upward jumps) that

$$(q-1)\theta(e^x-1)+1 > 0, \text{ for all } \theta \in [\theta_e - \epsilon, \theta_e + \epsilon],\ x \in \mathbb{R},$$

provided q is sufficiently small. Hence, $[\theta_e - \epsilon, \theta_e + \epsilon] \subset \mathrm{Dom}_q$ for sufficiently small q and by the previous lemma, there is a $q(\epsilon) > 1$ such that for all $1 < q < q(\epsilon)$

$$\Phi(q, \theta_e - \epsilon) < 0 < \Phi(q, \theta_e + \epsilon).$$

By continuity and monotonicity of $\Phi(q, \cdot)$, the root of $\Phi(q, \cdot)$, θ_q, is unique and satisfies $\theta_e - \epsilon < \theta_q < \theta_e + \epsilon$. Hence $|\theta_q - \theta_e| < \epsilon$ for all $1 < q < q(\epsilon)$. Finally, condition (4) is satisfied thanks to assumption (11), if θ_e is positive, or otherwise by the boundedness of upward jumps.
(ii) The proof of (ii) is similar but easier taking Proposition 2.4 into account. □

The convergence of the roots θ_q to θ_e is the main ingredient for the convergence of the measures. Indeed, with this result at hand, Theorem 2.6 can be proved along the lines presented in Niethammer (2007). Nonetheless we give a complete proof here that takes the additional difficulties stemming from possibly negative densities into account.

Theorem 4.4. *Under the conditions of Theorem 2.6, the asserted convergence holds in probability.*

Proof. The proof under the conditions of Theorem 2.6, (i) and (ii), can be given at the same time. Clearly q is to be read as $q(m)$ for item (ii).

Recall that $\theta_q \to \theta_e$ and therefore pointwise $\mathrm{eg}_q(x) \to e^{\theta_e(e^x-1)}$, as $q \downarrow 1$. By Yor's formula we can decompose

$$\mathcal{E}_T(\theta_q\sigma, \mathrm{eg}_q - 1) = \mathcal{E}_T(\theta_q\sigma, 0)\, \mathcal{E}_T(0, 1_{|\cdot|\leq 1}(\mathrm{eg}_q - 1))\, \mathcal{E}_T(0, 1_{|\cdot|>1}(\mathrm{eg}_q - 1)).$$

Obviously, we have P-almost surely,

$$\mathcal{E}_T(\theta_q\sigma, 0) \to \mathcal{E}_T(\theta_e\sigma, 0). \tag{19}$$

Moreover,

$$\mathcal{E}_T(0, 1_{|\cdot|>1}(\mathrm{eg}_q - 1)) = e^{-T\int_{|x|>1}[\mathrm{eg}_q(x)-1]\,\nu(dx)} \prod_{s\leq T} 1_{|\Delta\check{X}_s|>1}\mathrm{eg}_q(\Delta\check{X}_s).$$

The product consists pathwise of finitely many factors only, whence

$$\prod_{s\leq T} 1_{|\Delta\check{X}_s|>1}\mathrm{eg}_q(\Delta\check{X}_s) \to \prod_{s\leq T} 1_{|\Delta\check{X}_s|>1} e^{\theta_e(e^{\check{X}_s}-1)}, \quad P\text{-a.s.}$$

By (11), $|\mathrm{eg}_q(x)|$ is dominated on $\{x,\, |x| > 1\}$ by a ν-integrable function. Indeed, if $\theta_e < 0$, this follows as in the proof of Lemma 4.2. For $\theta_e > 0$ and q sufficiently small, we obtain,

$$|\mathrm{eg}_q(x)| \leq e^{\theta_q(e^x-1)} \leq e^{(\theta_e+\delta)(e^x-1)}.$$

Hence, we get P-almost surely

$$\mathcal{E}_T(0, 1_{|\cdot|>1}(\mathrm{eg}_q - 1)) \to \mathcal{E}_T(0, 1_{|\cdot|>1}(e^{\theta(e^{\cdot}-1)} - 1)). \tag{20}$$

To treat the remaining factor, note that there is a $\epsilon > 0$ depending on θ_e such that $\mathrm{eg}_q(x) > \epsilon$ for all x, $|x| \leq 1$, provided q is sufficiently small. Thus,

$$\mathcal{E}_T(0, 1_{|\cdot|\leq 1}(\mathrm{eg}_q - 1)) = \exp\left\{\int_0^T \int_{|x|\leq 1} \log(\mathrm{eg}_q(x))\tilde{N}(dx, ds)\right\}$$
$$\times \quad \exp\left\{-T\int_{|x|\leq 1} [\mathrm{eg}_q(x) - 1 - \log(\mathrm{eg}_q(x))]\,\nu(dx)\right\} =: e^{(I)+(II)}$$

To justify dominated convergence for the term (II), note that by Taylor's theorem

$$\begin{aligned} \log(\mathrm{eg}_q(x)) &= (\mathrm{eg}_q(x) - 1) - \frac{1}{2\xi(x)^2}(\mathrm{eg}_q(x) - 1)^2, \\ \mathrm{eg}_q(x) &= 1 + \Big((q-1)\theta_q(e^{\eta(x)} - 1) + 1\Big)^{1/(q-1)} \theta_q(e^x - 1) \end{aligned}$$

with intermediate points $\xi(x)$, $\eta(x)$ between 1 and $\mathrm{eg}_q(x)$ and between 0 and x, respectively. Consequently, for all x, $|x| \leq 1$, and sufficiently small q

(independent of x),

$$|\mathrm{eg}_q(x) - 1 - \log(\mathrm{eg}_q(x))]| \le \frac{1}{2\epsilon^2}|\theta_e + \delta|^2 e^{4|\theta_e+\delta|}(e^x - 1)^2,$$

which is ν-integrable over $\{x, |x| \le 1\}$. Thus,

$$\int_{|x|\le 1} [\mathrm{eg}_q(x) - 1 - \log(\mathrm{eg}_q(x))]\,\nu(dx) \to \int_{|x|\le 1} [e^{\theta(e^x-1)} - 1 - \theta(e^x - 1)]\,\nu(dx).$$

Concerning (I), we obtain from the isometry property of the stochastic integral,

$$\begin{aligned} & E\left[\left|\int_0^T \int_{|x|\le 1} [\log(\mathrm{eg}_q(x)) - \theta_e(e^x - 1)]\,\tilde{N}(dx, ds)\right|^2\right] \\ \le\ & T \int_{|x|\le 1} |\log(\mathrm{eg}_q(x)) - \theta_e(e^x - 1)|^2\,\nu(dx). \end{aligned}$$

The right hand side converges to zero by dominated convergence, since the integrand can be estimated for sufficiently small q by $2(|\theta_e| + \delta)^2(e^x - 1)^2$. We, thus, obtain, in probability,

$$\mathcal{E}_T(0, 1_{|\cdot|\le 1}(\mathrm{eg}_q - 1)) \to \mathcal{E}_T(0, 1_{|\cdot|\le 1}(e^{\theta(e^{\cdot}-1)} - 1)). \tag{21}$$

Combining (19)–(21) yields the assertion thanks to Yor's formula. □

Finally, we generalize from convergence in probability to $L^r(\Omega, P)$-convergence for some $r > 1$.

Proof of Theorem 2.6. Again, we treat the cases (i) and (ii) at same time and think of q as $q(m)$ under the set of conditions (ii). In view of the previous theorem it suffices to show that there are $r, q_0 > 1$ such that the family

$$\left(\left|\mathcal{E}_T(\theta_q\sigma, \mathrm{eg}_q - 1) - \mathcal{E}_T(\theta_e\sigma, e^{\theta_e(e^x-1)})\right|^r\right)_{1<q\le q_0}$$

is uniformly integrable.

Define $\kappa = \kappa(\mu) = 2\mu/(2\mu - 1)$ for $\mu \in \mathbb{N}$. Choosing μ sufficiently large and $1 < q \le q_0$ sufficiently small, we get

$$\int_{\mathbb{R}_0} |\mathrm{eg}_q^\kappa(x) - 1 - \kappa(\mathrm{eg}_q(x) - 1)|\,\nu(dx) \le K(\kappa) \tag{22}$$

for some constant $K(\kappa)$ independent of q. Indeed, as in the previous proof $1_{|x|>1}|\mathrm{eg}_q(x)|^\kappa$ is dominated by an ν-integrable function for sufficiently small κ and $1 < q \le q_0$. A straightforward application of Taylor's theorem shows that $1_{|x|\le 1}|\mathrm{eg}_q^\kappa(x) - 1 - \kappa(\mathrm{eg}_q(x) - 1)|$ is dominated by an ν-integrable function as well. Hence, (22) follows and implies that

$$\begin{aligned} & \mathcal{E}_T^\kappa(\theta_q\sigma, \mathrm{eg}_q - 1) = \mathcal{E}_T(\kappa\theta_q\sigma, \mathrm{eg}_q^\kappa - 1) \\ \times\ & \exp\left\{T\left(\frac{\kappa^2 - \kappa}{2}\theta_q^2\sigma^2 + \int_{\mathbb{R}_0} \mathrm{eg}_q^\kappa(x) - 1 - \kappa(\mathrm{eg}_q(x) - 1)\,\nu(dx)\right)\right\}. \end{aligned}$$

As the stochastic exponential of a Lévy martingale is a martingale and since $\kappa = 2\mu/(2\mu - 1)$, we have for all $1 < q \leq q_0$

$$\begin{aligned} & E\left[\left|\mathcal{E}_T(\theta_q\sigma, \mathrm{eg}_q - 1)\right|^\kappa\right] = E\left[\mathcal{E}_T^\kappa(\theta_q\sigma, \mathrm{eg}_q - 1)\right] \\ \leq\ & \exp\left\{T\left(\frac{\kappa^2-\kappa}{2}\theta_q^2\sigma^2 + \int_{\mathbb{R}_0} \left|\mathrm{eg}_q^\kappa(x) - 1 - \kappa(\mathrm{eg}_q(x) - 1)\right| \nu(dx)\right)\right\} \\ \leq\ & \tilde{K}(\kappa). \end{aligned}$$

In view of (11) we can assume without loss of generality that μ is large enough such that

$$\begin{aligned} & E\left[\left|\mathcal{E}_T(\theta_e\sigma, e^{\theta_e(e^x-1)})\right|^\kappa\right] \\ =\ & \exp\left\{T\left(\frac{\kappa^2-\kappa}{2}\theta_e^2\sigma^2 + \int_{\mathbb{R}_0} \left|e^{\kappa\theta_e(e^x-1)} - 1 - \kappa(e^{\theta_e(e^x-1)} - 1)\right| \nu(dx)\right)\right\} \\ <\ & \infty. \end{aligned}$$

Combining these two estimates, the de la Vallé-Poussin criterion yields the uniform integrability for all $1 < r < \kappa$. □

5. Some Consequences for Portfolio Optimization

In this section we collect some consequences for the portfolio optimization problem

$$\arg\max\{E(u_{2m}(X));\ X \in \Theta^{(2m),\tilde{x}}\}$$

with respect to the utility function $u_{2m}(x) = -(1 - \frac{x}{2m})^{2m}$, where

$$\Theta^{(2m),\tilde{x}} = \left\{X \in L^{2m}(\Omega, \mathcal{F}_T, P):\ \exists\vartheta \in \mathcal{A}^{(2m)} \text{ s.t. } X = \tilde{x} + \int_0^T \vartheta_u dS_u\right\}.$$

Recall that the set of admissible strategies $\mathcal{A}^{(2m)}$ was defined in Section 3. Our findings complement earlier results by Kohlmann and Niethammer (2007) and Niethammer (2007).

(1) The verification procedure for the q-optimal signed martingale measure in Section 3 implies that, under condition C_q^-, the optimal portfolio for the above optimization problem is given by

$$\begin{aligned} \vartheta_t^{(2m)} =\ & -\frac{2m-\tilde{x}}{2m-1}\mathcal{E}_t((q-1)\theta_q'\sigma, (q-1)\theta_q'(e^{\cdot} - \mathbf{1})) \\ & \times \exp\left\{t(q-1)\theta_q'\left(-\beta + \int_{\mathbb{R}_0^n}(e^x - \mathbf{1} - x1_{\|x\|\leq 1})\nu(dx)\right)\right\}\theta_q'\mathbf{S}_{t-}^{-1}. \end{aligned}$$

(2) It is straightforward to check that, under the assumptions of Theorem 2.6, $\vartheta^{(2m)}$ converges to

$$\vartheta_t^{(\infty)} = -\theta_e S_{t_-}^{-1}$$

uniformly on $[0,T]$ in probability, as m tends to infinity, thanks to Theorem 4.3. Then we obtain from the continuity of the stochastic integral

$$\lim_{m\to\infty}\sup_{0\le t\le T}\left|(x+\int_0^t \vartheta_u^{(2m)}dS_u)-(x+\int_0^t \vartheta_u^{(\infty)}dS_u)\right|$$

in probability, hence the convergence of the value processes.

Moreover, a direct calculation, making use of the semimartingale decomposition of S and Theorem 2.5, yields

$$\int_0^T \vartheta_u^{(\infty)}dS_u \quad = \quad -\log\mathcal{E}(\theta_e\sigma, e^{\theta_e(e^{\cdot}-1)}-1)+const.$$

By martingale considerations under the minimal entropy measure the constant must coincide with the relative entropy of the minimal martingale measure with respect to P and consequently $x+\int_0^T \vartheta_u^{(\infty)}dS_u$ is the optimal payoff of the exponential utility problem.

Hence, we have derived convergence of the utility maximization problem with utility $u^{(2m)}$ to the exponential utility problem even in situations, where the q-optimal martingale measures need not be equivalent.

Finally note that the optimal portfolio $\vartheta^{(\infty)}$ for the exponential utility problem can already be found in Kallsen (2000).

(3) Suppose that S is a one-dimensional (non-compensated) exponential Poisson process with jump height 2 instead of 1. As S is increasing, there will be arbitrage and, hence the exponential utility maximization problem has infinity as value. Nonetheless condition $C^-_{q(m)}$ is satisfied for all $m\in\mathbb{N}$ and a direct calculation shows that $\theta_{q(m)}=-(2m-1)$. Hence, the portfolio optimization problem with utility $u^{(2m)}$ has an optimal solution, although the model admits arbitrage (this comes from the fact that $u^{(2m)}(x)$ is not increasing for large x). Clearly, in this case $\theta_{q(m)}$ diverges to minus infinity.

Appendix A. Proofs of Propositions 2.2 and 2.4

Throughout the appendix we may assume that the set of equivalent martingale measures is non-empty and P is not a martingale measure. This immediately implies that $\sigma\neq 0$ or $\nu(\mathbb{R}\setminus\{0\})\neq 0$. (Otherwise the stock is strictly increasing or strictly decreasing depending on the sign of b.)

We start with the proof of Proposition 2.2.

Proof of Proposition 2.2. (i) Assume that C_q holds for some $q>1$, and hence the q-optimal equivalent martingale measure exists. Moreover, suppose that, for all $\theta>0$,

$$\int_{x\ge 1} e^{\theta e^x}\nu(dx)=\infty \tag{23}$$

and the MEMM exists. Then, due to Theorem 2.5 (ii), Φ_e has a zero θ_e on its domain Dom_e. Since, by (23), $\mathrm{Dom}_e=(-\infty,0]$, and P is not a martingale measure, we get $\theta_e<0$. Hence, by Lemma 4.1, we have $\Phi(q,0)>0$, and consequently the zero θ_q of $\Phi(q,\cdot)$ is negative. As the upward jumps are not bounded, thanks to (23), we have a contradiction to C_q^+.

(ii) Suppose that C_q holds for some q, (6) fails, and upward jumps are unbounded. Then we can again conclude from Lemma 4.1 that θ_q is negative contradicting C_q^+ as in (i). □

It remains to give the proof of Proposition 2.4.

Proof of Proposition 2.4. $\Rightarrow$: Suppose $C^-_{q(m)}$ holds for some fixed m, and abbreviate $q = q(m)$. Then, by definition,

$$\int_{x>1} |(((q-1)\theta_q(e^x-1)+1)^{\frac{q}{q-1}})-1-q(((q-1)\theta_q(e^x-1)+1)^{\frac{1}{q-1}}-1)|\nu(dx) < \infty.$$

Note that $2m = q/(q-1)$. Hence, for large x the term e^{2mx} dominates in the above integrand and we may conclude that (7) holds.

$\Leftarrow$: Suppose that condition (7) holds for some fixed m. Again we write q instead of $q(m)$. Recall that $\mathrm{Dom}_q = \mathbb{R}$ thanks to Lemma 4.1, (iii). Now, in view of Lemma 4.1, (i), it suffices to show that

$$\lim_{\theta\downarrow-\infty} \Phi(q,\theta) = -\infty \quad \text{and} \quad \lim_{\theta\uparrow\infty} \Phi(q,\theta) = \infty. \tag{24}$$

As noted at the beginning of the appendix, the existence of an equivalent martingale measure implies that $\sigma \neq 0$ or $\nu(\mathbb{R}\setminus\{0\}) \neq 0$. We first treat the case $\sigma \neq 0$. By a direct calculation, as in Niethammer (2007),

$$\theta\Phi(q,\theta) \geq -|\theta| \cdot |\int_{\mathbb{R}} (e^x - \mathbf{1}) - x1_{|x|\leq 1}\nu(dx) - \beta| + \sigma^2|\theta|^2.$$

Dividing by θ implies (24) immediately. If $\sigma = 0$, then $\nu(\mathbb{R}\setminus\{0\}) \neq 0$. Note that, for every given x, $(e^x - \mathbf{1})((q-1)\theta'(e^x - \mathbf{1}) + 1)^{\frac{1}{q-1}}$ is monotone in θ and converges to $\pm\infty$ for $\theta \to \pm\infty$, since $1/(q-1) = 2m-1$. Hence, the monotone convergence theorem yields (24) in the latter case. Obviously, condition (7) ensures that (4) holds. □

References

Choulli, T., Stricker, C., and Li, J. (2007), Minimal Hellinger Martingale Measures of Order q, Finance Stoch., forthcoming.

Cont, R. and Tankov, P. (2004), Financial Modeling with Jump Processes, Chapman & Hall: Boca Raton, Fl.

Esche, F. and Schweizer, M. (2005), Minimal Entropy Preserves the Lévy Property: How and Why, Stochastic Process Appl., 115, 299-237.

Fujiwara, T. and Miyahara, Y. (2003), The Minimal Entropy Martingale Measures for Geometric Lévy processes, Finance Stoch. 7, 509-531.

Grandits, P. and Rheinländer, T. (2002), On the Minimal Entropy Martingale Measure, Ann. Probab., 30, 3, 1003-1038.

Hubalek, F. and Sgarra, C. (2006), Esscher Transforms and the Minimal Entropy Martingale Measure for Exponential Lévy Models, Quantative Finance, 6, 2, 125-145.

Jeanblanc, M., Klöppel, S., and Miyahara, Y. (2007), Minimal F^Q-Martingale Measures For Exponential Lévy Processes, Ann. Appl. Probab., forthcoming.

Kallsen, J. (2000), Optimal Portfolios for Exponential Lévy processes, Math. Meth. Oper. Res., 51, 357-374.

Kohlmann, M. and Niethammer, C.R. (2007), On Convergence to the Exponential Utility Problem, Stoch. Proc. Appl. forthcoming.

Kohlmann, M. and Xiong, D. (2007), The p-Optimal Martingale Measure when there Exist Inaccessible Jumps. J. Pure Appl. Math., forthcoming.

Kunita, H. (2004), Representation of Martingales with Jumps and Applications to Mathematical Finance, in: *Stochastic Analysis and Related Topics in Kyoto 2004, In honour of Kiyosi Itô*, Advanced Studies in Pure Mathematics, 41, Mathemetical Society of Japan.

Luenberger, D. G.(1969), Optimization by Vector Space Methods, John Wiley & Sons, New York.

Mania, M., Santacroce, M., and Tevzadze, R. (2002), A Semimartingale Backward Equation related to the p-Optimal Martingale Measure and the Minimal Price of Contingent Claims, in: Stoch. Processes and Related Topics, Stochastic Monographs 12, London: Taylor and Francis.

Niethammer, C.R. (2007), On Convergence to the Exponential Utility Problem with Jumps, Stoch. Anal. Appl., forthcoming.

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