Relation

between

L^q-Optimality, Exponential Control, and Entropy

in

Risk Management

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Introduction

Apart from the theoretical interest in exponential utility functions, there is an economic motivation for the use of this special type of utility functions. Optimizing the investment decisions at a certain endpoint T of an investor with initial wealth x can be described by maximizing the expected exponential utility of a terminal value Y_T of a wealth process Y:

$$V_{exp,\xi}(x) = \max E(1 - e^{-\alpha(x+Y_T-\xi)}),$$

where ξ represents a financial obligation which the investor faces in T. Under certain assumptions the issue of finding an optimal terminal value of the exponential problem was completely solved, including contingent claims, in Delbaen et al (2002) and Kabanov and Stricker (2002), for a class of wealth processes different from the one we use. Variants of the concept appeared before, see Remark 2.1 in Delbaen et al (2002) for further references. Moreover, a BSDE (backward stochastic differential equations) approach can be found in Rouge and El Karoui (2000) and Hu et al (2004), the second one does not use any dual relationship, see both papers for further references. The exponential utility function belongs to a class of utility functions, we later call utility functions of type II. Schachermayer (2001) completely solved the utility maximization problem for this class of utility functions. Again, the setting is different from ours and an explicit portfolio is not established, see section 3.2.2 for a comparison to our approach.

From an economic point of view, but also from high mathematical interest in control theory, we like to know the way how to obtain this optimal terminal value! We are interested in the special form of the portfolio. Up to some very special cases, see e.g. Delbaen et al (2002) and Rouge and El Karoui (2000), there are no results on this important question. Fortunately, results on iso-elastic utility functions ($x^p,\ p > 1, p = 2r, r \in \mathbb{N}$) exist. The aim of this thesis is to tie a connection between both problems to form a basis to derive a portfolio for the exponential problem using the already established portfolio in the iso-elastic case. In Theorem 5.2.4, we present a complete relation between various types of martingale measures (dual problem), the iso-elastic, and the exponential problem. To our best knowledge, this approach is entirely new and might play an important role towards further research concerning the exponential utility problem.

The chosen market consists of n stocks, modelled by an n-dimensional semimartingale S, and one constant bond $(S, 1)$, i.e. the bond is a numéraire. The market is discounted. That causes no further restrictions, see e.g. Leitner (2001). Allowable strategies have to be self-financing (adding and removing money from the portfolio is not allowed) and have to satisfy an integrability condition. We only allow for $p-$integrable trading strategies (see Definition 1.2.1). Furthermore, we consider wealth processes where we allow to withdraw money from the portfolio. This is modelled by an increasing right-continuous consumption component such that the terminal value of the consumption process is L^p-integrable. The optimal terminal wealth is then in L^p. This enables us to use methods from convex analysis to derive the optimal terminal wealth in chapter 3. The idea is very old, but in this setting the exact procedure is described in this thesis for the first time, to the best of our knowledge. The set, we optimize over in the original problem, is a set of wealth processes. Wealth processes, measurable functions in time and of random events, are difficult to treat. Before, in chapter 2 we therefore transfer the dynamic optimization problem to a static problem (an optimization over a class of random variables - constant in time). The problem is solved in Delbaen and Schachermayer (1996b) without a consumption component (here Theorem 2.2.4). This does not matter for exponential utility functions, since they are strictly increasing. However, iso-elastic functions are not everywhere increasing, so we adopt a result from El Karoui and Quenez (1995) and use an optional decomposition theorem from Föllmer and Kabanov (1998) (here 2.2.8) to establish the result including a consumption component (Theorem 2.2.10).

The static problem can now be written as a convex (resp. concave) optimization problem - a constraint problem over elements in L^p. We know from convex analysis: If we find a saddle point of the corresponding Lagrange functional, we also have found a solution of the original problem. This saddle point problem leads to a dual problem. We translate the results in Luenberger (1969) to stochastic analysis and develop a method to solve the saddle point problem. The stochastic dual problem then optimizes over the positive cone generated by the L^q-densities of martingale measures. To solve the problem it therefore makes sense to have a closer look at different measures. We explain the concepts of the so-called minimal martingale measure, the minimal entropy measure, and the $q-$optimal measures ($q > 1$) and discuss their connections. This is essential since the minimal entropy measure is the dual solution to the exponential problem and the $\frac{2m}{2m-1}-$optimal measure is the dual solution to the 2m-th member of the approximating sequence of the exponential utility function, whereas the minimal martingale measure has a very nice form. Under certain assumptions, e.g. in a Brownian model with deterministic volatility and drift, the measures are identical. Furthermore, Mania et al (2003a) and Mania et al (2003b) introduce a representation of the $q-$optimal measures and the minimal entropy measure, respectively, using BSDEs. Applying these results, Mania et al (2005) show under certain assumptions on the class of martingale measures the L^1-convergence of the densities of the $q-$optimal measures to the density of the minimal entropy measure for $q \downarrow 1$. That means the solution of the 2m-th dual problem converges to the solution of the exponential problem.

Next, we modify the iso-elastic problem and consider the sequence $u_p(x) = -(1 - \frac{x}{p})^p,\ p = 2m$. Using the approach suggested in chapter 3, we solve this problem (5.7). The optimal terminal wealth is then a function of the dual solution - the density of the $q-$ optimal measure, where $q = \frac{2m}{2m-1}$ is conjugate to $p = 2m$. The convergence of the $\frac{2m}{2m-1}-$ optimal measure to the minimal entropy measure - the dual solution of the exponential problem - then implies the main result of this thesis: The convergence of the optimal terminal wealth and the corresponding value function of the $2m$-th problem to the exponential one. This is mathematically rather interesting, but - as mentioned in the beginning - this can also be seen as a first important step to establish an explicit portfolio of the exponential problem - a result of enormous practical importance.

The thesis is organized as follows. At the beginning a summary in German is given. Chapter 1 introduces notation and some important results. Further, the semimartingale market model and the specialized Brownian model are explained. Finally, we give a short economic overview mainly explaining the concept of optimizing expected utility. A more detailed explanation can be found in the appendix. Chapter 2 presents the main problems and reformulates the dynamic problem to a static problem. Chapter 3 embeds the stochastic problem in the framework of convex analysis and suggests a method to solve the convex problem. Examples for the case $\xi \equiv 0$ in the complete and incomplete case are given. A first convergence result of the solution of the partial sums of the exponential utility function to the solution of the exponential problem is established. We conclude the chapter with some comments to the problem $\xi \neq 0$. Chapter 4 discusses different solutions of dual problems. Chapter 5 contains the main result of this thesis, the convergence of the optimal terminal values of the 2m-th problem. Finally, we summarize and draw a conclusion. The appendix contains some economic explanations.

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Zusammenfassung

Die vorliegende Diplomarbeit beschäftigt sich mit der Nutzenoptimierung von Vermögenswerten zu einem vorher spezifizierten Endzeitpunkt (reines Investitionsproblem). Des Weiteren werden Forderungstitel zum Endzeitpunkt bewertet (Bewertungsproblem) bzw. zusätzlich miteinbezogen (gemischtes Problem). Optimal sind Strategien, die den erwarteten Nutzen zum Endzeitpunkt maximieren. Diese Annahme hat Vor- und Nachteile. Sie wird deshalb im Anhang kurz diskutiert. Wir betrachten vor allem isoelastische $-x^{2r}$, $r \in \mathbb{N}$ und exponentielle Nutzenfunktionen $-e^{-\alpha x}$, $\alpha > 0$. Letztere beschreiben ein Investitionsproblem mit etwaigen Bewertungkomponenten in idealer Weise und sind daher für die Ökonomie von besonderer Wichtigkeit. Jedoch ist es bis jetzt, abgesehen von kleineren Spezialfällen, nicht möglich gewesen, eine optimale Anlagestrategie explizit für diesen exponentiellen Fall herzuleiten. Für den isoelastischen Fall ist dies in Bürkel (2004) beschrieben. Das Ziel dieser Arbeit ist es nun, einen Zusammenhang zwischen beiden Problemen herzustellen. Die vollständige Beziehung zwischen optimalen Martingalmaßen (duales Problem), dem isoelastischen und exponentiellen Problem wird in Kapitel 5 dargelegt (siehe Satz 5.2.4). Dieses Ergebnis ist, soweit uns bekannt, gänzlich neu und könnte bei der Berechnung eines optimalen Portefeuilles und der weiteren Erforschung von exponentiellen Nutzenfunktionen ein wichtiges Werkzeug sein. Wir bedienen uns hierbei bekannter Methoden der konvexen Analysis: Wir übersetzen diese in den Kontext der stochastischen Analysis und entwickeln eine neue Lösungsmethode im vorliegenden Setting, die für viele Nutzenfunktionen anwendbar ist. Dies geschieht in Kapitel 3.

Die Investoren können zwischen n verschiedenen Aktien und einer konstanten risikolosen Anlage wählen $(S, 1)$, wobei S durch ein n-dimensionales Semimartingal beschrieben wird. Um Methoden der konvexen Analysis einsetzen zu können, wählen wir unser Modell so, dass die Vermögenswerte zum Endzeitpunkt Zufallsvariablen im reflexiven Banachraum L^p sind, siehe Kapitel 1. Davor müssen wir das gegebene dynamische in ein geeignetes statisches Problem überführen, d.h.: Wir wollen den erwarteten Nutzen des Endwertes eines Vermögensprozesses maximieren. Wir optimieren über stochastische Prozesse, Funktionen in der Zeit und im Zufall. Dieses Problem kann in eine Optimierungsaufgabe unter Nebenbedingung über Zu-

fallsvariablen umgewandelt werden. Für unser spezielles Modell und für ansteigende Nutzenfunktionen wurde dies in Delbaen and Schachermayer (1996b) gezeigt. Um den oben erwähnten Zusammenhang herzustellen, benötigen wir aber auch Lösungen für isoelastische Funktionen. Diese sind aber nicht überall ansteigend. Die Umwandlung des dynamischen in ein statisches Problem für diesen Typ von Nutzenfunktionen ist zwar bekannt, aber nicht für die von uns erlaubten Strategien. In Kapitel 2 führen wir daher einen Beweis für p-integrierbare Strategien, siehe Satz 2.2.10.

Zu dem statischen Problem unter Nebenbedingung können wir nun die Lagrangefunktion aufstellen. Aus der konvexen Analysis wissen wir Folgendes: Finden wir einen Sattelpunkt der Lagrangefunktion, ist der zweite Teil des Sattelpunkts auch eine Lösung des ursprünglichen Problems. Um das Sattelpunktproblem zu lösen, gehen wir auf ein duales Problem über - auf das der Lagrangemultiplikatoren. Diese bestehen aus einem Produkt einer nichtnegativen reellen Zahl und einer L^q-Dichte eines Martingalmaßes. Wir betrachten also ein Problem über Martingalemaße. Zu diesem Zweck beschäftigen wir uns in Kapitel 4 mit verschiedenen Martingalmaßen. Wir stellen das minimale Martingalmaß, das Entropie-minimale und das q-optimale Martingalmaß vor und geben einige nützliche Charakterisierungen. Wir beschreiben, wann die Dichten der Maße fast sicher übereinstimmen. Weiter führen wir Bedingungen an, wann die q-optimalen Maße gegen das Entropie-minimale für q gegen 1 konvergieren (L^1- Konvergenz). Letzteres wurde in Mania et al (2005) bewiesen. Unter Benutzung der Resultate in selbigem Papier führen wir den Beweis leicht abgewandelt in Kapitel 5 noch einmal.

Das Entropie-minimale Maß ist die duale Lösung zum exponentiellen Problem. Das $\frac{2m}{2m-1}$-optimale Maß ist die Lösung zum Problem mit der Nutzenfunktion: $u_{2m}(x) = -(1 - \frac{x}{2m})^{2m}$. Die dualen Lösungen der Folge u_{2m} konvergieren also gegen die dualen Lösungen des Exponentialproblems. Dies ist nicht überraschend, da $u_{2m}(x)$ gegen $-e^{-x}$ konvergiert, wenn m gegen unendlich strebt. Unter Benutzung dieses Resultats is es uns gelungen die L^1- Konvergenz des primären Problems zu zeigen (Kapitel 5). Bei Übereinstimmung der Maße erhalten wir sogar fast sichere Konvergenz.

Die Nutzenfunktionen u_{2m} sind Modifikationen der isoelastischen Nutzenfunktionen. Man kann nun versuchen, das schon bekannte Portefeuille des isoelastischen Falls für den Fall der u_{2m} umzuschreiben, um dann durch das von uns gezeigte Konvergenzresultat ein Portefeuille für den exponentiellen Fall zu entwickeln.

Zum Schluss sei erwähnt, dass wir hauptsächlich den Fall ohne Forderungstitel betrachten, jedoch entwickeln wir am Ende der Kapitel 3 und 5 einige Ideen zu deren Behandlung.

Contents

Chapter 1

Toolbox

In this chapter, we introduce the notation and state some theorems important for this thesis. We explain the used market model and the concept of maximizing expected utility.

1.1 Definitions and Important Theorems

We introduce the following notation:

Definition 1.1.1. (Time and stopping times) Deterministic time points are always written in Latin letters, usually denoted by t, varying within the time interval $[0, T]$, where T is finite and deterministic as well. Processes are always defined on $[0, T]$. $\mathcal{T}$ denotes the set of all stopping times. Its elements are written in Greek letters, usually denoted by τ.

We define the following classes of processes:

Definition 1.1.2. Let $T < \infty$, and all the following classes be sets of processes defined on $[0, T]$. We denote the space of all

1. adapted $\mathbb{R}^n$ processes by $\mathbb{A}_n$.

2. (local), [continuous] martingales M with $E^{\frac{1}{p}}(\sup_{t\in[0,T]} |M_t|^p) < \infty$ by $\mathbb{M}^{p,[c]}_{(loc)}$, $\mathbb{M}^{1,[c]}_{0,(loc)} =: \mathbb{M}^{[c]}_{0,(loc)}$.

3. (local), [continuous] martingales M with $E^{\frac{1}{p}}(\sup_{t\in[0,T]} |M_t|^p) < \infty$ and $M_0 = 0$ by $\mathbb{M}^{p,[c]}_{0,(loc)}$.

4. [continuous] uniformly integrable martingales M with $E^{\frac{1}{p}}(|M_T|^p) < \infty$ by $\mathcal{U}^{p,[c]}$, $\mathcal{U}^{1,[c]} =: \mathcal{U}^{[c]}$.

5. predictable one-dimensional processes H with

$$E(\int_0^T H_t^2 d\langle M\rangle_t)^p)^{\frac{1}{p}} < \infty \text{ for } M \in \mathcal{U}.$$

by $L^p(M)$. If this is only valid locally, we write $L^p_{loc}(M)$.

Further, we introduce some terms from $BMO-$theory:

Definition 1.1.3. A martingale N is in BMO_p, if there exists a constant $C > 0$, such that:

$$\forall t \in [0, T] \; E(|N_T - N_t|^p|\mathcal{F}_t) \leq C.$$

The smallest constant satisfying this inequality is called the BMO norm and is denoted by $\|M\|_{BMO_p}$. If N is continuous then the sets are equal for all p, in this case the set is called BMO.

In this thesis, we exclusively work with special semimartingales rather than semimartingales (in the sequel, we use the term semimartingale as a shortcut for special semimartingales):

Definition 1.1.4. A special semimartingale is a process with the unique representation:

$$S_t = S_0 + M_t + A_t,$$

where M is a local martingale with $M_0 = 0$ and A is a predictable finite variation process with $A_0 = 0$. We denote the space of all special semimartingales by $\mathcal{S}$.

For uniqueness see Jacod (1979) page 29/30. Note, if not otherwise posted, in the unique decomposition $S = S_0 + M + A$ M always denotes the martingale part and A always the part of bounded variation. We want to equip $\mathcal{S}$ with a norm, therefore we define the quadratic variation and quadratic covariation, taken from Protter (2004):

Definition 1.1.5. (Quadratic variation and covariation) Let X, Y be in $\mathcal{S}$. The quadratic variation process of X, denoted by $[X] = ([X]_t)_t$, is defined by

$$[X] = X^2 - 2\int X_- dX,$$

where $(X_-)_s = \lim_{u\to s, u<s} X_u$. The quadratic covariation of X, Y, or bracket process of X is defined by:

$$[X, Y] = XY - \int X_- dY - \int Y_- dX,$$

$[X, Y]^c$ denotes the path by path continuous part of $[X, Y]$.

$[X, Y]$ is bilinear and symmetric and $[X, X] = [X]$. Further, the following holds (see Protter (2004) Theorem 22/23 and Corollary 1 p.66):

Theorem 1.1.1. *The quadratic variation process of X is an RCLL (right-continuous with left-limits), increasing, adapted process. Further, let p_n be any sequence of random partitions tending to identity ($0 = \tau_0^n \leq \tau_1^n \leq ... \leq \tau_{k_n}^n$ and τ_i^n stopping times), then:*

$$X_0^2 + \sum_i (X_{\tau_{i+1}^n} - X_{\tau_i^n})^2 \to [X]$$

converging in ucp (uniform convergence on compacts in probability).
$[X, Y]$ is a semimartingale and has paths of finite variation on compacts. Moreover, we again have: Let p_n be any sequence of random partitions tending to identity ($0 = \tau_0^n \leq \tau_1^n \leq ... \leq \tau_{k_n}^n$ and τ_i^n stopping times), then:

$$X_0 Y_0 + \sum_i (X_{\tau_{i+1}^n} - X_{\tau_i^n})(Y_{\tau_{i+1}^n} - Y_{\tau_i^n})^2 \to [X, Y]$$

converging in ucp.

Note, convergence is even almost surely, if the stopping times are deterministic time points, see Jacod (1979) pp. 171/172. Before, we go on to the next theorem, we introduce some notation:

$$\Delta X_t = X_t - X_{t-}, \ t > 0$$

if the left limit X_{t-} of X_t exists. $X_{0-} = 0$ and $\Delta X_0 = X_0$. The next theorem ties a connection to the usual definition of $\langle \cdot, \cdot \rangle$, defined below:

Theorem 1.1.2. *Let M be a local martingale with continuous paths that are not everywhere constant. Then $[M]$ is not the constant process M_0^2, and $M^2 - [M]$ is a continuous local martingale. Moreover, if $[M]_t = 0$ for all t then $M_t = 0$ for all t. Let M be, in addition, locally square-integrable, but not necessarily continuous and N another locally square-integrable local martingale. Then $[N, M]$ is the unique adapted RCLL-process A with paths of finite variation on compacts satisfying the following two properties:*

1. *$NM - A$ is a local martingale*

2. *$\Delta A = \Delta N \Delta M$, $A_0 = M_0 N_0$*

If M is continuous then $\Delta A = 0$.

This corresponds to the following definition of the compensator process $\langle \cdot, \cdot \rangle$ in Jacod (1979) page 34:

Definition 1.1.6. (Compensator) Let $M \in \mathbb{M}^2_{loc}$ then the unique increasing predicable process $\langle M \rangle$, such that:

$$M^2 - \langle M \rangle \in \mathbb{M}_{0,loc},$$

is called the compensator of M. Let further $N \in \mathbb{M}^2_{loc}$, we define $\langle N, M \rangle$ as:

$$\langle N, M \rangle = \frac{1}{4}(\langle N + M \rangle - \langle N - M \rangle)$$

or equivalently $\langle N, M \rangle$ is the unique predicable process such that $NM - \langle N, M \rangle$ is a local martingale vanishing in zero.

Note, $\langle \cdot, \cdot \rangle$ is a predicable process, whereas $[\cdot, \cdot]$ is RCLL. However, if $M, N \in \mathbb{M}^c_{loc}$ there are also in $\mathbb{M}^2_{loc}$. $\langle M, N \rangle$ is well-defined, continuous and equals $[M, N] = [M, N]^c + M_0 N_0$. Since we work with special semimartingales, we can uniquely decompose every $X \in \mathcal{S}$ such that:

$$X = M + A + J, \; J_t = \sum_{0 \le s \le t} \Delta X_s$$

and M a continuous local martingale and A a continuous finite variation process with $M_0 = A_0 = 0$. The unique continuous local martingale is denoted by X^c, see Protter (2004) p.221 for details. So the connection between the compensator and the quadratic covariation is as follows. Let $X, Y \in \mathcal{S}$,

$$\begin{aligned} [X, Y] &= \langle X^c, Y^c \rangle + \sum_{0 \le s \le t} \Delta X_s \Delta Y_s \\ &= \langle X^c, Y^c \rangle + X_0 Y_0 + \sum_{0 < s \le t} \Delta X_s \Delta Y_s \end{aligned}$$

See (2.24) in Jacod (1979) and page 62 and Theorem 28 in Protter (2004). In the sequel, we set

$$S(\Delta X \Delta Y) := \sum_{0 \le s \le t} \Delta X_s \Delta Y_s.$$

If now $X, Y \in \mathcal{S}$ and X is continuous, we have:

$$\begin{aligned} [X, Y] &= \langle X^c, Y^c \rangle + X_0 Y_0 + \sum_{0 < s \le t} \Delta X_s \Delta Y_s \\ &= \langle X^c, Y^c \rangle + X_0 Y_0 = \langle X, Y \rangle \end{aligned}$$

or for $M, N \in \mathbb{M}^c_{loc}$:

$$\begin{aligned} [M, N] &= \langle M^c, N^c \rangle + M_0 N_0 + \sum_{0 < s \le t} \Delta M_s \Delta N_s \\ &= \langle M^c, N^c \rangle + M_0 N_0 = \langle M, N \rangle \end{aligned}$$

We mostly work with continuous martingales, both brackets coincide. Before we go on with normed spaces of semimartingales, we state Corollary 2 from Protter (2004):

Corollary 1.1.3. *(Integration by Parts) Let X, Y be two semimartingales. Then XY is a semimartingale and*

$$XY = \int X_- dY + \int Y_- dX + [X, Y].$$

We define two normed spaces of semimartingales - $\mathcal{H}_n^p(P)$ and $\mathcal{S}_n^p(P)$:

Definition 1.1.7. Let $1 \leq p \leq \infty$, then

$$\mathcal{H}_n^p(P) = \{Y \in \mathcal{S} | \mathbb{R}^n - \text{valued and } \|Y\|_{\mathcal{H}_n^p} < \infty\}. \tag{1.1}$$

where

$$\|Y\|_{\mathcal{H}_n^p} = \|Y_0 + \tilde{M} + \tilde{A}\|_{\mathcal{H}_n^p} = \|\text{diag}([\tilde{M}]_T)^{\frac{1}{2}} + \int_0^T |d\tilde{A}_t|\|_{L^p} + \|Y_0\|_{L^p(P)}.$$

$\mathcal{S}_n^p(P)$ is the class of p-integrable semimartingales ($p \in [1, \infty)$):

$$\mathcal{S}_n^p(P) = \{Y \in \mathcal{S} | \mathbb{R}^n - \text{valued with } \|Y\|_{\mathcal{S}_n^p} < \infty\}. \tag{1.2}$$

where $\|Y\|_{\mathcal{S}_n^p} = \| \sup_{t \in [0,T]} |Y_t| \|_{L^p}$. $\mathcal{S}_1^p(P) =: \mathcal{S}^p(P)$ and $\mathcal{H}_1^p(P) =: \mathcal{H}^p(P)$.

Note, in both cases we usually leave out the index n, if it does not lead to confusions. By the Burkholder-Davis-Gundy-inequality,

$$(E(\sup_{0 \leq \tau \leq T} N_\tau)^p)^{\frac{1}{p}} \leq C(p)(E[N]^{\frac{p}{2}})^{\frac{1}{p}}, \; C(p) \in (0, \infty), \; N \in \mathbb{M}_{loc} \tag{1.3}$$

(set without loss of generality $Y_0 = 0$), we have:

$$\begin{aligned} E^{\frac{1}{p}} \sup_{t \in [0,T]} (|Y_t|^p) &= E^{\frac{1}{p}} \sup_{t \in [0,T]} ((|\int_0^t d\tilde{A}_s + \tilde{M}_t|^p) \\ &\leq CE^{\frac{1}{p}}(|\int_0^T |d\tilde{A}_t| + \text{diag}[\tilde{M}]_T^{\frac{1}{2}}|^p) \\ &\leq C\|Y\|_{\mathcal{H}_n^p}, \; C \in (0, \infty) \end{aligned}$$

Hence, $\mathcal{H}^p(P) \subset \mathcal{S}^p(P)$. Note, the Burkholder-inequality holds in the reverse direction:

$$(E[N]^{\frac{p}{2}})^{\frac{1}{p}}, \leq c(p)(E(\sup_{0 \leq \tau \leq T} N_\tau)^p)^{\frac{1}{p}}, \; N \in \mathbb{M}_{loc} \; c(p) \in (0, \infty)$$

So if the considered process is a local martingale in $\mathcal{S}^p(P)$ it is also in $\mathcal{H}^p(P)$.

We next define orthogonality:

Definition 1.1.8. Two local martingales C, B are strongly orthogonal, denotes by $C \perp B$ if CB is a local martingale vanishing in zero. If both martingales are continuous, therefore in $\mathbb{M}^2_{loc}$, this is the case if and only if $\langle B, C\rangle = 0$ by Corollary 1.1.3.

Strong orthogonality of $N, M \in \mathbb{M}^2$ implies that N, M are weakly orthogonal, i.e. $E(M_T N_T) = 0$. We further state Theorem (4.27) taken from Jacod (1979):

Theorem 1.1.4. *(Galtchouk-Kunita-Watanabe-decomposition) Let $M, N \in \mathcal{H}^2$ then there exists $H \in L^2(M)$ and a unique element $\tilde{N} \in \mathcal{H}^2$ with $\tilde{N} \perp M$ such that*

$$N = \int H dM + \tilde{N}$$

See Jacod (1979) p. 126 for a proof. The last Theorem can be extended easily to the case $p \geq 2$ since $\mathcal{H}^p \subset \mathcal{H}^2$. For the case $p \in (1, 2)$ this turns out to be a slightly more complicated, see Jacod (1979) Chapter IV. For a definition of the stochastic integral with respect to a martingale of predictable processes, we refer to Jacod (1979) or Protter (2004). For a multidimensional consideration see Ansel and Stricker (1993b). The main result is the following, let $M \in \mathcal{S}^p_n$ and continuous, and taking the closure in $\mathcal{S}^p$ then:

$$\overline{\left\{\sum_{i=1}^n \int_0^\cdot N^i dM^i : \ \forall N^i \in L^p(M^i)\right\}} = \left\{\int_0^\cdot N dM | \ N \in L^p(M)\right\}$$

where

$$L^p(M) = \{N | \mathbb{R}^n - \text{valued, predictable with } \|N\|_{L^p(M)} < \infty\}$$

with $\|N\|_{L^p(M)} := (E(\int_0^T N' d\langle M\rangle_t N)^{\frac{p}{2}})^{\frac{1}{p}}$

Next, we introduce the concept of relative entropy:

Definition 1.1.9. The relative entropy $H(Q|P)$ of a probability measure Q with respect to a probability measure P is defined as

$$H(Q|P) = \begin{cases} E_P(\frac{dQ}{dP} \log \frac{dQ}{dP}), & \text{if } Q \ll P \\ \infty, & \text{otherwise} \end{cases}$$

The relative entropy admits non-negative values and is zero if and only if the two measures coincide. It is a measure of distance between two distributions. However, it is not a metric. In general, $H(P|Q) = H(Q|P)$ does not hold!

Finally, some general notations: ' denotes the transposition. p and q are usually conjugate, that means $\frac{1}{p} + \frac{1}{q} = 1$. We always use the notation log for the natural logarithm instead of ln. BSDE is an abbreviation of backward stochastic differential equation. For a definition and a good overview of the theory of BSDEs in the Brownian case see Kohlmann (2003). The concept of BSDEs goes back to Bismut (1973) for linear generators and it was generalized in Chitashvili (1983) and Pardoux and Peng (1990). Moreover, Chitashvili (1983) introduced BSDEs in a general semimartingale setting, see also Chitashvili and Mania (1987) and Chitashvili and Mania (1996). The stochastic exponential $\mathcal{E}_t(M)$ of a continuous local martingale M is defined as follows:

$$\mathcal{E}_t(M) = \exp(M_t - \frac{1}{2}[M]_t) = \exp(M_t - \frac{1}{2}\langle M \rangle_t).$$

1.2 Market Models

1.2.1 General Semimartingale Models

In this section, from Delbaen et al (2002) and Grandits and Krawczyk (1998) we mainly take over a market model in a semimartingale setting used throughout this thesis without further mentioning it. The model is specialized to a Brownian setting in section 1.2.2.

In general, we work in a semimartingale model. If not otherwise posted all results are valid for the following model: Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T \in (0, \infty)$ a time horizon, and $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ a filtration satisfying the usual conditions, i.e. right-continuity and completeness (all sets of $\bigcup \mathcal{F}_t = \mathcal{F}_T$ with measure zero are also elements of $\mathcal{F}_0$). This enables us to use right-continuous with left-limits (RCLL) versions for all $(P, \mathbb{F})-$semimartingales representing our stocks. We further assume that all process on $[0, T]$ have sufficient properties such that all martingales are uniformly integrable. We use special semimartingales, i.e. a Doob-Meyer- decomposition holds, see Definition 1.1.4. For simplicity we just use the term semimartingale. Throughout this thesis a continuous (in some cases this will be relaxed) $\mathbb{R}^{n+1}-$valued $(P, \mathbb{F})-$semimartingale $(S, 1)$, where $S = (S_t)_{t \in [0,T]}$ with unique decomposition $S = S_0 + M + A$, represents a vector of n risky and 1 a riskless asset with discounted price constant at 1, i.e. the riskless asset serves as a numeraire. We further impose the following assumption:

Assumption 1.2.1. *S is locally bounded, i.e. there exists a sequence of stopping times (τ_m) increasing to infinity almost surely such that for each $m \geq 1$ $(S_{t \wedge \tau_m})_t$ is bounded.*

Since we have chosen a finite time horizon, we have:

Lemma 1.2.1. *Every continuous process on $[0, T]$ with $T < \infty$ is locally bounded.*

That follows from the fact that a continuous function is bounded on a compact interval. So the following assumption is stricter than 1.2.1:

Assumption 1.2.2. *S is a process with continuous paths.*

We discuss some cases, when only assumption 1.2.1 holds. However, we fix assumption 1.2.2 as given, if not otherwise posted.

We are interested in optimizing a portfolio consisting of n risky assets and one riskless investment. We invest an initial capital in the portfolio, call it initial wealth, at the beginning. Then we look at the portfolio how it behaves in time while restructuring it without adding further money, i.e. we only allow for self-financing strategy. Such a strategy is fully determined by its initial wealth x and the number of shares N_t^i of stock i for $i = 1, ..., n$, we hold in t. The amount of money invested in the bond is then given by:

$$\pi_0(t) = x - \sum_{i=1}^{n} N_t^i S_i(t).$$

Summarizing: We invest according to the strategy N and put the remaining money in the riskless asset or finance the strategy by selling the riskless asset, respectively. The vector $N = (N^1, ..., N^n)$ is in $L^n(S)$, where $L^n(S)$ is the set of all $\mathbb{F}-$predictable, S-integrable $\mathbb{R}^n-$ valued processes. This is sufficient to define the following gain $(G(N))$ and wealth process $(V(x, N))$, when using this self-financing strategy with initial wealth x for $t \in [0, T]$:

$$G_t(N) = \int_0^t NdS \tag{1.4}$$

$$V_t(x, N) = x + \int_0^t NdS \tag{1.5}$$

Note, the bond S^0 is 1 for all $t \in [0, T]$, so for all t, we have that $\Delta^t S^0 = 0$. Next, we can also allow for additional possibility of consumption, however the strategies N stay in the same form and are always assumed to be self-financing. Further, the requirement that $N \in L^n(S)$ is not enough to exclude arbitrage opportunities or doubling strategies. That is a very lax assertion. We first have to make precise what an arbitrage opportunity means and then define a appropriate class of allowed strategies. There are several possibilities and they might lead to different results in portfolio optimization. Several notions of arbitrage-free markets exists. Some of them correspond to sets of martingale measures. So later, we just assume that a certain set of martingale measures is not empty, which can be proven to be equivalent to a certain non-arbitrage-condition. Further, we require that our trading strategies satisfy certain integrability conditions. Wealth processes are martingales, when using these strategies and thus doubling strategies are excluded, see Lemma

1.2.5 or use the inequality of Burkholder-Davis-Gundy. Section 3.2.2 (Utility Functions of Type II) then shows how this corresponds to the restriction of a finite credit line and the exclusion of doubling strategies.

We will optimize over the set of final values of wealth processes by applying results from convex analysis to this set. We therefore work with reflexive Banach spaces. So we specify a class of trading strategies $\mathcal{A}^p$, such that the final values of the wealth processes are L^p-integrable:

$$\mathcal{G}^p(x) := \{Y_T | Y \in \mathcal{W}(x)\} \subset L^p(P) \tag{1.6}$$

where

$$\mathcal{W}(x) := \{Y | Y_t = x + \int_0^t N dS, N \in \mathcal{A}^p\} \subset \mathcal{S}^p(P) \subset \mathcal{H}^p(P) \tag{1.7}$$

is the class of all wealth process generated by the class $\mathcal{A}^p$ of L^p-trading strategies (see below), i.e. Y_T can be hedged by initial wealth $x \in \mathbb{R}$ and trading strategy N.

To establish (1.6) and (1.7), we use the following class of trading strategies, as in Grandits and Krawczyk (1998) and Grandits and Rheinländer (2002a):

Definition 1.2.1. (L^p/p-integrable -trading strategies) The set of L^p-trading strategies is defined as follows:

$$\mathcal{A}^p = L^p(M) \cap L^p(A) \tag{1.8}$$

where

$$\begin{aligned} L^p(M) &= \{N \in \mathbb{P}^n | \, \|N\|_{L^p(M)} < \infty\} && (1.9)\\ &\quad \text{where } \|N\|_{L^p(M)} := (E(\int_0^T N' d\langle M\rangle_t N)^{\frac{p}{2}})^{\frac{1}{p}} && (1.10)\\ L^p(A) &= \{N \in \mathbb{P}^n | \, \|N\|_{L^p(A)} < \infty\} \\ &\quad \text{where } \|N\|_{L^p(A)} := (E(\int_0^T |N' dA_t|)^p)^{\frac{1}{p}} && (1.11) \end{aligned}$$

and $\mathbb{P}^n$ the set of all predictable $\mathbb{R}^n$-valued processes.

The stochastic integral is well-defined (see Jacod (1979)) and by the triangular inequality the inclusions in (1.7) hold since

$$\begin{aligned}
\|Y\|_{\mathcal{H}^p} &= \|Y_0 + \int NdM + \int NdA\|_{\mathcal{H}^p} \\
&\leq \|\int NdM\|_{\mathcal{H}^p} + \|\int NdA\|_{\mathcal{H}^p} + \|Y_0\|_{\mathcal{H}^p(P)} \\
&= (E(\langle \int NdM\rangle_T^{\frac{p}{2}})^{\frac{1}{p}} + \left(E\left(\int_0^T |N'dA_t|\right)^p\right)^{\frac{1}{p}} + x \\
&= E(\int_0^T N'd\langle M\rangle_t N)^{\frac{p}{2}})^{\frac{1}{p}} + \left(E\left(\int_0^T |N'dA_t|\right)^p\right)^{\frac{1}{p}} + x < \infty
\end{aligned}$$

Suppose we add the possibility to remove money from the market via consumption, but without rewarding this consumption, i.e. utility of consumption is zero. Considering the class $\mathcal{W}$ is then enough for increasing utility functions, since for every superhedging strategy (a self-financing strategy and a consumption strategy which replicates the claim) there exists a hedging strategy obtaining a better or equal terminal wealth, i.e. adding the possibility to remove money from the market via consumption does not yield a higher terminal wealth. However, in the case of partly non-increasing function this makes sense without rewarding consumption. We introduce:

$$\mathcal{W}_C(x) = \{Y | Y_t = x + \int_0^t NdS - C_t, N \in \mathcal{A}^p,\ C \in \mathcal{K}^p\} \tag{1.12}$$

and $\mathcal{A}^p$ as before and $\mathcal{K}^p$ the class of increasing right-continuous processes with $C_T \in L^p$.

On account of the duality of L^p and L^q Definition 1.2.1 naturally leads to the following "dual" classes of measures with respect to S:

Definition 1.2.2. (Measures) We define the following classes of measures:

1. The space of all density processes of signed L^q-integrable local martingale measures for S is defined as:

$$\mathcal{D}_S^q = \{Z \in \mathcal{U}^q | E(Z_T) = 1, SZ \in \mathrm{M}_{loc}\} \tag{1.13}$$

This can be identified with the set of its densities:

$$\mathcal{M}_{S,Z}^q = \{Z_T | Z \in \mathcal{D}_S^q\} \subset L^q(P) \tag{1.14}$$

The corresponding signed local martingale measures are denoted by:

$$\mathcal{M}_S^q = \{Q | dQ = Z_T dP,\ Z_T \in \mathcal{M}_{S,Z}^q\} \tag{1.15}$$

2. The space of all density processes of absolutely continuous L^q-integrable local martingale measures for S is defined as:

$$\mathcal{D}_a^q = \{Z \in \mathcal{U}^q | E(Z_T) = 1,\ Z_T \geq 0,\ SZ \in \mathrm{M}_{loc}\} \tag{1.16}$$

This can be identified with the set of its densities:

$$\mathcal{M}_{a,Z}^q = \{Z_T | Z \in \mathcal{D}_a^q\} \subset L^q(P) \tag{1.17}$$

The corresponding local martingale measures are denoted by:

$$\mathcal{M}_a^q = \{Q | dQ = Z_T dP,\ Z_T \in \mathcal{M}_{a,Z}^q\} \tag{1.18}$$

3. The space of all density processes of equivalent continuous L^q-integrable local martingale measures for S is defined as:

$$\mathcal{D}_e^q = \{Z \in \mathcal{U}^q | E(Z_T) = 1,\ Z_T > 0,\ SZ \in \mathrm{M}_{loc}\} \tag{1.19}$$

This can be identified with the set of its densities:

$$\mathcal{M}_{e,Z}^q = \{Z_T | Z \in \mathcal{D}_e^q\} \tag{1.20}$$

The corresponding local martingale measures are denoted by:

$$\mathcal{M}_e^q = \{Q | dQ = Z_T dP,\ Z_T \in \mathcal{M}_{e,Z}^q\} \subset L^q(P) \tag{1.21}$$

Note, if different stocks are considered, we add a bracket of S, i.e. $\mathcal{M}(S)$. Further, the following holds:

Lemma 1.2.2. *Under assumption 1.2.1, a probability measure Q is in $\mathcal{M}_a^1$ ($\mathcal{M}_e^1$) if and only if $Q \ll P$ ($Q \approx P$) and $E_Q(h'(S_\tau - S_\sigma)) = 0$ for all stopping times $\sigma \leq \tau \leq T$ so that S_τ is bounded and for all $\mathbb{R}^n$-valued bounded $\mathcal{F}_\sigma$-measurable random variables h.*

Signed local martingale measures were firstly introduced in Müller (1985). $\mathcal{D}_S^q$ and $\mathcal{D}_e^q$ correspond to different notions of arbitrage. The strong law of one price (SLP) holds if and only if $\mathcal{D}_S^q \neq \emptyset$, see Leitner (2001) p.14. $\mathcal{D}_e^q \neq \emptyset$ is equivalent to the NFLVR-no-arbitrage-condition (no-free-lunch-with-vanishing-risk) and implies also the SLP, since $\mathcal{D}_e^q \subset \mathcal{D}_S^q$, for details see Delbaen and Schachermayer (1994), Delbaen and Schachermayer (1995a), and Delbaen and Schachermayer (1996b). All classes are important in utility maximization problems. In section 4.2, we state results from Leitner (2001) that under very general assumptions (S is only assumed to be an adapted process) $\mathcal{D}_S^q \neq \emptyset$ implies the unique existence of the solution of an expected terminal utility maximization problem (for a certain class of utility functions). The solution exists in the sense that it is in the L^p-closure of final values of integrals of simple self-financing hedging strategies. This generalizes an approach in Delbaen and Schachermayer (1996a). However, this does not ensure that there exists a hedging strategy to reach this value. We therefore work with special semimartingales satisfying the above integrability conditions, instead of a general adapted process. We further impose the assumption 1.2.4 below to ensure that it is enough to work with $\mathcal{M}_e^q$. With this assumption, it turns out (see Theorem 4.1.13), that the $q-$optimal measure (the solution of $\min_{Z\in\mathcal{M}_{Z,S}^q} E(Z^q)$) is in $\mathcal{M}_{Z,e}^q$. Furthermore, the set $\mathcal{G}^p$ is closed, so that we can optimize over this set and find the optimum within this class. To exclude arbitrage opportunities and to get a assertion like "Strong Law of One Price" or "No Free Lunch with Vanishing Risk", respectively, we assume:

Assumption 1.2.3. *The space $\mathcal{M}_e^q$ is not empty.*

This turns out, except excluding arbitrage opportunities, to be a very helpful assumption in solving optimization problems. The dual problem, here an optimization over standardized measures (corresponding cone), has a solution in the above space up to standardization.

Before we conclude this section, we come back to the set of allowable trading strategies. In Delbaen and Schachermayer (1996b), they first consider simple p-admissible strategies and define the corresponding integral. The closure of the space of these integrals $\mathbb{K}^p(x)$ is then equal to the closure of $\mathcal{G}^p(x)$, see Grandits and Rheinländer (2002a) or see Lemma 1.2.3. For $\mathbb{K}^p(x)$, we have a nice hedging result for L^p claims, which then also holds for $\mathcal{G}^p(x)$, if it is already closed. The closedness is true under assumption 1.2.4 (Reverse Hölder inequality, see Theorem 1.2.4). So using this assumption, Delbaen and Schachermayer (1996b) show the mentioned hedging result: Every $f \in L^p$ satisfying $E_Q(f) = x$ for every $Q \in \mathcal{M}_a^q$ is in $\mathcal{G}^p(x)$ and f can be replicated with initial wealth x (see Theorem 2.2.4). This motivates the following definition, proposed by Delbaen and Schachermayer (1996b):

Definition 1.2.3. An $\mathbb{R}^d$-valued predictable process H is called a simple $p-$admissible integrand for S, if it is a linear combination of processes of the form:

$$H_f = f 1_{(\tau_1,\tau_2]},$$

where $\tau_1 \leq \tau_2$ finite stopping times smaller or equal to T and $f \in L^\infty(\Omega, \mathcal{F}_{\tau_1}, P)$.

Note, we assume that S is continuous (locally bounded) and therefore locally in every L^p. Delbaen and Schachermayer (1996a) do not require that the process is continuous, they have to assume L^p-integrability and therefore they introduce the notion p-admissible. We further use the following notation:

$$\mathrm{V}^p(x) = \{x + (H\cdot S)_T : H - \text{simple and } p\text{-admissible } \} \tag{1.22}$$

Lemma 2.1. in Grandits and Rheinländer (2002a) give the following connection between $\mathcal{G}^p$ and V^p:

Lemma 1.2.3. *Under assumption 1.2.2 and $\mathcal{M}_e^q \neq \emptyset$, then*

$$\overline{\mathcal{G}^p(x)} = \overline{\mathrm{V}^p(x)} =: \mathbb{K}^p(x)$$

Further, we introduce the Reverse Hölder inequality $R_p(Q)$:

Definition 1.2.4. A process Z satisfies the Reverse Hölder inequality $R_q(Q)$, if there exists a $C(q) > 1$ such that

$$\sup_{\tau \in \mathcal{T}} E_Q \left(\left| \frac{Z_T}{Z_\tau} \right|^q |F_\tau \right) < C(q). \tag{1.23}$$

We propose the following assumptions:

Assumption 1.2.4. *A) All $(F, P)-$ local martingales are continuous.*
B) There exists an equivalent martingale measure Q such that its density process satisfies the reverse Hölder inequality $R_{p_0}(P)$ for some fixed $p_0 > 1$.

Note, these assumptions imply that $\mathcal{M}_q^e \neq \emptyset$ and $\langle \cdot, \cdot \rangle$ coincide with $[\cdot, \cdot]$ when applying to martingales. Grandits and Krawczyk (1998) prove the following result:

Theorem 1.2.4. *Let S be a continuous semimartingale, $1 < p < \infty$, and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following assertion are equivalent:*

1. *There exists a martingale measure Q in $\mathcal{M}_q^e$ and for all $x \in \mathbb{R}$ $\mathcal{G}^p(x)$ is closed in $L^p(P)$.*
2. *There exists a martingale measure Q in $\mathcal{M}_q^e$ and for some $x \in \mathbb{R}$ $\mathcal{G}^p(x)$ is closed in $L^p(P)$.*
3. *There exists a martingale measure Q in $\mathcal{M}_q^e$ that satisfies the Reverse Hölder inequality $R_q(P)$*

So $\mathcal{G}^p(x) = \overline{\mathrm{V}^p(x)}$ under assumption 1.2.4, i.e. wealth processes generated by a L^p- hedging strategy are also in the closure of wealth process generated by simple-hedging strategies.

Besides pure utility maximization, we treat (approximate) hedging problems. A contingent claim is an $\mathcal{F}_T$-measurable random variable satisfying some integrability conditions, if not otherwise posted $\xi \in L^p(P)$. We say that contingent claim is attainable ξ, if there exists an allowed strategy N and initial wealth x such that the terminal value of the generated process is equal to the contingent claim ξ. A common characterization is that the set of martingale measures is a singleton. We assume that both assertion are equivalent and therefore define:

Definition 1.2.5. We say a market consisting of a bond and n stocks represented by the stochastic process $(S,1)$ is arbitrage-free if $\mathcal{M}_a = \mathcal{M}_e^q \neq \emptyset$ and we say that this market is complete if $\mathcal{M}_e^q = \{Q\}$.

Remark 1.2.1. Assumption 1.2.2 and 1.2.3 are standing assumptions. Assumption 1.2.4 is not always necessary in every case, so it is listed in any case if it is needed.

1.2.2 Brownian Market Models

In this section, we specialize the model defined in section 1.2.1. The martingale part is a Brownian martingale and the part of bounded variation is supposed to be absolutely continuous with respect to the Lebesgue measure. We again have one non-risky asset and n risky assets - stocks. Risky means that the stock is driven by a drift and a noise term. All components will be continuous. The non-risky asset does not have this noise term. However, the drift of the non-risky asset, i.e. the risk free rate, can be random! The following model was first introduced by Samuelson (1965) and further developed by Merton (1971)/ Merton (1973) and Black and Scholes (1973). In the original model market (σ, μ, r) data were assumed to be constant. In the model below, we allow for random, possibly unbounded coefficients.

We start with the noise term. Let $W_t' = (W_t^1, ..., W_t^d)', t \in [0,T],\ T \in (0,\infty)$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_t$ the augmented filtration generated by this Brownian motion. So the filtration satisfies the usual conditions. We continue with the riskfree asset. To stay consistent, it should be equal to 1. Despite, we introduce the general model and afterwards switch to the discounted model: We require that the interest rate r is $\mathcal{F}_t$-progressively measurable and

$$\int_0^T |r_t| dt < \infty \quad P - a.s. \tag{1.24}$$

Assuming (1.24) the bond can be equivalently defined by:

$$B_t = \exp\left[\int_0^t r_s ds\right] \Leftrightarrow dB_t = r_t B_t dt,\ B_0 = 1 \tag{1.25}$$

So to fit in the general semimartingale model, we set $r \equiv 0$. The n risky investments S^i are modelled by

$$S_t^i = s_0^i \exp\left[\int_0^t \left(\mu_s^i - \frac{1}{2}\sum_{j=1}^d (\sigma_s^{i,j})^2\right) ds + \sum_{j=1}^d \int_0^t (\sigma_s^{i,j})^2 dW_s^j\right], \tag{1.26}$$

where μ_s^i and $\sigma_s^{i,j}$ are $\mathcal{F}_t-$ progressively measurable and $s_0^i \geq 0$. To ensure that the integrals are well-defined, we require that:

$$\sum_{i=1}^n \sum_{j=1}^d \int_0^T |\mu_t^i| + |\sigma_s^{i,j}|^2 dt < \infty \quad P - a.s. \tag{1.27}$$

By Itô's formula (1.26) is the solution of:

$$dS_t^i = \mu_t^i S_t^i dt + \sum_{j=1}^{d} \sigma_t^{i,j} S_t^i dW_t^j,\ S_0^i = s_0^i \tag{1.28}$$

We further assume:

Assumption 1.2.5. *1. $d \geq n$*

2. The diffusion matrix $\sigma = (\sigma^{i,j})_{i=1,\ldots,n,j=1,\ldots,d}$ has $P \otimes \lambda$-a.s. maximal rank.

3. There exists a solution of $\mu_t - r_t 1 = \sigma_t \theta_t$ satisfying $\int_0^T \|\theta_t\|^2 dt < \infty$ P-a.s. and the corresponding Girsanov functional $Z_T = \mathcal{E}_T(-\int \theta dW)$ has expectation 1.

4. $\exists \epsilon > 0\ \sigma\sigma'(t) \geq \epsilon I_{n\times n}$ $P \otimes \lambda$-a.s..

2. is implied by 4.. In several cases the following assumption is very useful:

Assumption 1.2.6. *The coefficients r, μ, σ are uniformly bounded in ω and t.*

If assumption 1.2.6 is fulfilled, then $\bar{\theta} = \sigma'(\sigma\sigma')^{-1}(\mu - r)$ is bounded. So assumption 1.2.5.3 is satisfied since Novikov's condition holds. $\mathcal{M}_e^q$ is not empty for every $q \in [1, \infty)$ and the market is arbitrage free.

Next we come to the wealth process and the corresponding portfolios, analogously to $\mathcal{A}^p$:

Definition 1.2.6. We require an allowed portfolio $(\pi^i)_i' = (N^i S^i)'$ to fulfill the following integrability condition:

$$E((\int_0^T |\pi'\sigma|^2 dt)^{\frac{p}{2}})^{\frac{1}{p}} + E((\int_0^T |\pi'(\mu - r_t 1)|dt)^p)^{\frac{1}{p}} < \infty\ P - a.s. \tag{1.29}$$

and to be self-financing, i.e.:

$$dY_t = \frac{Y_t - \sum_{i=1}^n \pi_t^i}{B_t} dB_t + \sum_{i=1}^{n} \frac{\pi_t^i}{S_t^i} dS_t^i \tag{1.30}$$

In analogy to the semimartingale case, it is used that:

$$\pi = \mathbf{S}N$$

where

$$\mathbf{S}_{n\times n} = \begin{bmatrix} S^{(1)} & & 0 \\ & \ddots & \\ 0 & & S^{(n)} \end{bmatrix}$$

In the sequel when discussing portfolios, we always mean self-financing ones. The corresponding wealth process is then of the form:

$$dY_t = [r_t Y_t + \sum_{i=1}^{n}(\mu_t^i - r_t)\pi_t^i]dt + \sum_{j=1}^{d}\sum_{i=1}^{n}\sigma_t^{i,j}\pi_t^i dW_t^j,\ Y_0 = y \qquad (1.31)$$

where $r \equiv 0$. So π is an allowed portfolio if and only if $N \in \mathcal{A}^p$ as defined in 1.2.1 and the wealth processes correspond.

Let $\mathcal{K}$ denote the class of increasing right-continuous processes C with $C_0 = 0$ and $C_T < \infty$ almost surely. We obtain the following theorem:

Lemma 1.2.5. *The density Z_T of every equivalent martingale measure Q in $\mathcal{M}_e$ can be represented as a Girsanov-functional*

$$Z_T = \exp\left\{-\int_0^T \theta_s' dW_s - \frac{1}{2}\int_0^T \|\theta_s\|^2 ds\right\} \qquad (1.32)$$

where the process θ is a solution of $\sigma\theta = \mu$. If $Z_T \in L^q(P)$ and we only consider p-integrable portfolios $\pi \in \mathcal{A}^p$ with respect to P then all martingale measures $Q \in \mathcal{M}_e^q$ are even in $\mathcal{M}_{e,R}^q$, i.e. the wealth process is a martingale under Q, not only a local martingale. If we include an increasing consumption process $C \in \mathcal{K}$, then the process $dY = \pi\sigma dW_Q - dC$ is a Q-supermartingale.

Proof. We start with an arbitrary equivalent measure local martingale measure Q. Q is equivalent to P, there exists a density $Z_T > 0$. Set $Z_t = E(Z_T|F_t)$, by the martingale representation theorem with have the following representation:

$$Z_t = Z_0 + \int_0^t \phi_s dW_s = 1 + \int_0^t \phi_s Z_s Z_s^{-1} dW_s, \phi \in L^1(P) \qquad (1.33)$$

Since $Z_T > 0$, also $E(Z_T|F_t) > 0$, Z_t^{-1} exists for all t. The solution of (1.33) is:

$$Z_t = \exp\{-\int_0^t -\phi_s Z_s^{-1} dW_s - \frac{1}{2}\int_0^t \|\phi_s\|^2 Z_s^{-2}\} \qquad (1.34)$$

Define $\theta_s = -\phi_s Z_s^{-1}$, then by Girsanov's theorem we have

$$W_t^Q = W_t + \int_0^t \theta_s ds \qquad (1.35)$$

is then a Brownian motion under Q. Suppose the consumption process is zero. Then θ_s solves $\sigma_s\theta_s = \mu$, because a wealth process $Y \in \mathcal{W}(x)$ is a local $Q-$martingale. (If $\sigma_s\theta_s \neq \mu$, we have left a nonzero term of bounded variation.) We now prove that the wealth process is even a martingale: We have that

$$\left(E(\int_0^T |\pi'\sigma|^2 dt)^{\frac{p}{2}}\right)^{\frac{1}{p}} + \left(E(\int_0^T |\pi'(\mu - r_t 1)|dt)^p\right)^{\frac{1}{p}}$$

Define $g_t = \int_0^t |\pi'\sigma|^2 dt)^{\frac{p}{2}})^{\frac{1}{p}}$. So $E(g_t^p) < \infty$. Hence $g_t \in L^p(P)$. Since $Z_T \in L^q(P)$ we have by Hölder's inequality that $E(Z_T g_T) < \infty$ and therefore $g_T \in L^1(Q)$, hence $E_Q(\int_0^T |\pi'\sigma|^2 dt)^{\frac{1}{2}} < \infty$. According to Lemma 8.2.16 in Kohlmann (2003) (it is an application of the Burkholder-Davis-Gundy-inequality), we have that $\int_0^t \pi'\sigma dW_s^Q$ is $Q-$martingale . □

To conclude, we tie a connection to the terms arbitrage-free and complete market. Under assumption 1.2.5 1.-3. the Brownian Market is arbitrage free and it is complete if and only if $n = d$ ($\sigma\theta = \mu$ has exactly one solution), see Kohlmann (2003) Theorem 2.3.6 and 2.3.9. Note again, assumption 1.2.5 is always alleged. Assumption 1.2.6 is again not needed in every case.

1.3 Short Economic Overview

Individuals face different problems. The first group wants to invest a certain amount of money to obtain an optimal payoff at a certain time point T (pure investment problem). Another group just wants to know today a price of product (contingent claim) at time T (pricing problem). Most people face both of the above problems. They want to optimize their optimal payoff at T, but that also depends on the products they have to buy at T (mixed problem). The first question that arises is: What does optimal mean? This definitely depends on the risk and the expected return of the strategy. So the goal of this thesis is to optimize our investment decision according towards risk and return or even more general towards our preferences. This is a very vague assertion. The second question that arises is how to model a preference structure. How shall we model that investor 1 prefers investment A against investment B and a second investor 2 does it the other way around? This clearly depends on how risk-averse the investors are - on their preferences. Since this is not our topic, we answer this question in the appendix (section A.1). In addition the three mentioned problems are explained more detailed. However, we treat some aspects in this section. We briefly explain the meaning of a utility function and why we optimize the expected utility of our terminal wealth. We introduce several utility functions.

1.3.1 Preference Structure

In a stock market, we face various investments (call this set **A**) and we want to sort them according to a certain preference structure. Under several assumption to the economics, we can define an equivalence relation $\succeq$, where $A \succeq B$ means that we prefer A against B or are at least we are indifferent between both alternatives. Then we are able to assign every investment A in $\mathbf{A}$ a number $U(A)$ such that following holds:

$$\exists U : \mathbf{A} \to \mathbb{R}\ \forall A, B \in \mathbf{A} :\ U(A) \geq U(B) \Leftrightarrow A \succeq B,$$

see section (A.1). An investment, e.g. a stock, is a function in time and of random events. We therefore decide to consider the quality of our portfolio always at a certain endpoint T. However, it is possible that $U(A(T, \omega_1))$ is bigger than $U(B(T, \omega_1))$ for one state of the world ω_1. But this can turn around when we live in another state ω_2. These states are assigned with a certain probability, so this observation leads to the criterion of optimizing expected utility of the terminal wealth of the portfolio:

$$\sup_{X \in \mathcal{W}(x)} E(U(X_T)) \tag{1.36}$$

or when facing claims

$$V_{\xi,C}(x) = \sup_{X \in \mathcal{W}_C(x)} E(U(X_T - \xi)), \tag{1.37}$$

respectively. This definitely better than just optimizing the expected return of the portfolio, since expected utility accounts for risk. We take (1.36) and (1.37) as the now standing criterion to find an optimal portfolio. Appropriate utility functions and their interpretations are described in the next section.

1.3.2 Different Utility Functions

This section list several kinds of utility functions. We go into detail when working with them in the following chapters. We start with a typical class of utility functions. Typical since it is increasing and concave and negative wealth is not allowed. A type of utility function, we would look for when optimizing our personal long-life-investment decision:

1. Utility functions of type I: It contains function of the following form: $U : (0, \infty) \to \mathbb{R}$, strictly increasing, strictly concave and twice continuously differentiable with $U'(\infty) := \lim_{x\to\infty} U'(x) = 0$ and $U'(0+) := \lim_{x\downarrow 0} U'(x) = \infty$ (INADA-condition). Sometimes the following extension is considered: $U(x) = -\infty$ for $(-\infty, 0)$, $U(0) = 0$ to indicate that e.g. negative wealth must not happen.

2. Utility functions of type II: In contrast to type I, this class is defined on the whole real line. Again it is a finitely-valued, once or twice continuously differentiable, increasing, strictly concave function $U : \mathbb{R} \to \mathbb{R}$ with the following properties: $U'(\infty) := \lim_{x\to\infty} U'(x) = 0$ $U'(-\infty) := \lim_{x\to-\infty} U'(x) = \infty$.

3. Power utility functions (Iso-elastic functions with $0 < p < 1$) are defined as
$$U : [0, \infty) \to \mathbb{R}^+,\ U : x \mapsto x^p$$

4. Exponential utility with parameter $\alpha > 0$ is defined as
$$U : \mathbb{R} \to \mathbb{R},\ U_{exp}(x) = 1 - \exp(-\alpha x)$$

5. Approximating sequence of the exponential function
$$U : \mathbb{R} \to \mathbb{R},\ U_{2m}(x) = 1 - (1 - \frac{\alpha x}{2m})^{2m}$$

6. Iso-elastic utility functions $p > 1$ are defined as
$$U : \mathbb{R} \to \mathbb{R}^+, U : x \mapsto -|x|^p$$
We usually set $p = 2r$ and leave out $|\cdot|$.

All functions are strictly concave and increasing except the last two classes. 3. is a special case of 1. and 4. is a special case of 2.. The class of utility functions of type I and II are quite large. So to solve optimization problems, we will need further assumption. For type I we need that it has asymptotic elasticity at ∞ :
$$AE_\infty(U) := \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$$
and for type II in addition asymptotic elasticity at $-\infty$:
$$AE_{-\infty}(U) := \liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1$$
If a utility function satisfies both of these conditions, we say that it has reasonable asymptotic elasticity. An example is the exponential utility function. Power utilities ($p \in (0, 1)$) satisfy the first condition. Isoelastic functions with a parameter bigger than 1 do not satisfy this requirement. Nevertheless, solutions exists under some weak assumptions.

Chapter 2

Problem Formulation and its Reformulation

This chapter presents the main problem discussed in this thesis, see section 2.1. The posed problem is dynamic. To use methods from convex analysis, we need to reformulate the problem to a static problem. This is done in section 2.2.

2.1 Dynamic Problem

2.1.1 Hedging and Superhedging Strategies

In section 1.2.1, we have already touched the definition of hedging and superhedging strategies. We introduced the class of L^p-investment strategies $\mathcal{A}^p$ and the class of L^p-consumption processes $\mathcal{K}^p$. In the sequel a, not further specified, class of predictable investment strategies in $L(S)$ is denoted by $\mathcal{A}$ and a class of increasing right-continuous processes C with $C_0 = 0$ and $C_T < \infty$ almost surely - symbolizing consumption - by $\mathcal{K}$. In general, a hedging strategy for a contingent claim, mathematically an $\mathcal{F}_T-$measurable $L^p(P)-$integrable random variable, is an investment strategy such that the corresponding wealth process yields the same payoff as the contingent claim at T without consuming anything on $[0, T]$, i.e. without substracting any money from the portfolio. A superhedging strategy is a self-financing investment strategy and a consumption process such that the corresponding wealth process yields the same payoff at T as the contingent claim. We include consumption processes in model if a claim is not attainable with just investing in stocks and a riskfree investment. Note, the term superhedging strategy has nothing to do with idea that consumption should also have a positive influence on our satisfaction. This only plays a role in investment problems and is shortly discussed at the end of section 2.1.2 ((2.5), below).

We start explaining the terms hedging and superhedging strategies: We define:

Definition 2.1.1. A tuple $(N, C) \in \mathcal{A} \times \mathcal{K}$ is called a superhedging or superreplicating strategy with initial wealth $x \in \mathbb{R}$ for a contingent claim f if: There is a an $\mathcal{F}_t-$ measurable wealth process $Y^{x,N,C}$ corresponding to the superhedging strategy (N, C) so that

$$Y_t = x + \int_0^t N_s dS_s - C_t, \; Y_T = f \tag{2.1}$$

If $C_t \equiv 0$ for all t, we call N a hedging/replicating strategy.

In the Brownian setting using BSDEs (we set $\pi_i(t) = N_i(t)S_i(t)$) superhedging and hedging strategies are defined as follows:

Definition 2.1.2. A pair $(\pi, C) \in \mathcal{A} \times \mathcal{K}$ is called a superhedging or superreplicating strategy with initial wealth $x \in \mathbb{R}$ for a contingent claim f if, there is an $\mathcal{F}_t-$ progessively measurable wealth process $Y^{x,\pi,C}$ corresponding to the superhedging strategy (π, C) so that

$$dY_t = \pi_t' \mu_t dt - dC_t + \pi_t' \sigma_t dW_t, \; Y_T = f \tag{2.2}$$

and $E(\sup_{0 \leq t \leq T} |Y_t|^p) < \infty$.
If $C_t \equiv 0$ for all t, we call π a hedging/replicating strategy.

Note, in a Brownian model with bounded coefficients we do not need to assume that $E(\sup_{0 \leq t \leq T} |Y_t|^p) < \infty$, if $\mathcal{A} = \mathcal{A}^p$ and $\mathcal{K} = \mathcal{K}^p$ by Lemma 2.2.9.

We mainly use the trading strategies given in section 1.2.1. Section 3.2.2 (Utility Functions of Type II) additionally introduces some other trading strategies.

2.1.2 Problem Formulation

There are different kinds of utility maximization problems. As assumed in our model, all investors are always interested in maximizing their expected utility. But they might be in different scenarios. Some of them face a terminal pay-off ξ ($\mathcal{F}_T$-measurable and $L^p(P)-$ integrable random variable) and are primarily concerned with approximately hedging the claim to find a price. The initial wealth of the optimal wealth process, which gives the best approximate hedge, forms a substitute for the price. The second group just owns a certain amount of money and wants to invest this money optimally according to their preferences, a pure investment problem. A third group wants to maximizes its overall wealth, but also faces claims - a mixed problem. We introduce these different problems and discuss appropriate utility functions afterwards.

Different Economic Problems

We start with the pricing problem. Suppose, we start with an initial wealth of x. $V(x)$ denotes the value function of the problem, i.e. maximum expected utility we can achieve with an initial wealth of x. The first group and the last group deals with the following problem:

$$V(x)_{\xi,i} \equiv \sup_{Y\in\mathcal{W}_i(x)} E[U(Y_T-\xi)],\ x\in\mathcal{E} \tag{2.3}$$

where $\mathcal{W}_i$ is an appropriate class of wealth processes, $i=C$ stands for wealth processes with consumption otherwise we leave out this index. $\mathcal{E}$ is the set of admissible initial endowments. We are supposed to find a process $Y=(Y_t)_{t\le T}$ with initial wealth x, which maximizes the expected utility. We face a dynamic problem. Furthermore, we want to ensure that a trading (and consumption) strategy exist such that we can achieve or at least surpass the wealth process Y. So $\mathcal{W}_i$ shall be of the form $\mathcal{W}(x)=\{Y|Y_T = x+\int NdS, N\in\mathcal{A}\}$ or $\mathcal{W}_C(x)=\{Y|Y_T=x+\int NdS+C_t, N\in\mathcal{A},\ C\in\mathcal{K}\}$ and $\mathcal{A}$ a suitable class of trading strategies and $\mathcal{K}$ a class of right-continuous increasing consumption processes. We have already described two classes in section 1.2.1, $\mathcal{A}^p$ and $\mathcal{K}^p$. In section 3.2.2 (Utility Functions of Type II), we describe several other strategies. So using the definition of $\mathcal{W}$, we can equivalently write (2.3) as (independently of a specified trading strategy):

$$V(x)_{\xi,C} \equiv \sup_{(N,C)\in\mathcal{A}\times\mathcal{K}} E[U(Y_T^{(x,N,C)}-\xi)],\ x\in\mathcal{E} \tag{2.4}$$

where $Y^{(x,N,C)}$ is the wealth process which is generated by the initial wealth x, the portfolio process N, and the consumption process C.

Possible choices of utility functions for problem (2.3) and (2.4) are isoelastic utility functions with even $p>1$, (hedging problem) and exponential utility function (mixed problem).

The mentioned second group essentially treats the same problem with $\xi\equiv 0$. The choice of a suitable class of utility functions is different. One could use power utilities with $p\in(0,1)$ or also exponential utility functions, since there are increasing. In the sequel, we discuss the utility maximization problem in general, not specifying the utility function. However, we have to have in mind that the utility function is different. It makes no sense to find the best approximate hedge and use power utilities, since they are not defined on the negative real line and not in zero, which symbolizes a perfect hedge, we stick to the exponential and isoelastic ($p>1$ and even) utility functions. In spite, we can try to treat both problems - the pure utility maximization and the hedging problem - in the same way, i.e. it turns out that in a lot of cases it is sufficient to discuss the pure utility maximization problem. We give some examples in the section 3.2.3 when this is possible and only treat the case $\xi\equiv 0$ in the following chapters.

Different Economic Problems and their Utility function

Next, we discuss some interpretations and connections between the suggested problems and corresponding utility functions. Utility functions of type I are the ones, we would choose when discussing a pure optimization problem, since there are increasing, but the marginal utility is decreasing. That means more wealth yields a higher utility, but the more wealth we have the less worth is an additional unit of money or the smaller is the marginal utility. Power utilities are examples. They have an increasing utility functions so we only have to consider hedging strategies. Superhedging strategies (removing money from the portfolio - consumption - is allowed) always lead to the same or a lower utility value. Iso-elastic utility functions are not appropriate in this case. If superhedging strategies are not allowed, at a certain point it would punish to much wealth. A transformation might be reasonable when modelling waste disposal or warehouse costs. Even if superhedging strategies are allowed, the goal would be the maximum while consuming the remaining money. However, usually the isoleastic utility functions are used to model a hedging problem. Zero is the highest achievable value, symbolizing a perfect hedge. Consumption comes into account when having too much money or outperforming the claim, respectively. The exponential utility function models a mixture of both problems. If $x := X - \xi$ is zero, the utility function is also zero again symbolizing the claim is met. The difference to iso-elastic functions is outperforming the claim increases utility, but not as much as we lose when missing the claim at the same amount. This symbolizes that the main goal is still hedging the claim but not at all costs. The approximating sequence is a transformation of the iso-elastic case, but approaches the exponential case. If an approximation is possible for the solution, we discuss in chapter 5.2. Further, exponential utility functions do not depend on its initial wealth, but its approximating sequence, since it has a maximum, which we try to hedge (like in the iso-elastic case most visible in the quadratic case (mean-variance hedging)). However, this maximizer tends to infinity for n going to infinity, so that the problem -of switching to an optimal superhedging strategy- vanishes for large n and fixed initial wealth x.

Hedging versus Superhedging

Consider partial sums of the exponential utility function or a mean-variance problem ((2.3) and $U(x) = -x^2$) in a complete market. If the wealth is too high, we hedge the claim and consume the remaining money. So the superhedging strategy is actually determined by the hedging strategy and consuming the initial wealth minus the needed initial wealth to hedge the claim. Hence, it can happen that we outperform the claim, if it is not allowed to consume the remaining wealth. We are forced to invest the remaining money, we do not meet the claim and even a safe investment will not maximize our expected utility. Hence, a superhedging strategy can be better than all admissible hedging strategies. Consequently, the supremum over all superhedging strategies is strictly bigger than the one over all hedging strategies. This is a consequence of the missing monotony of $U(x) = -x^2$. If the main goal is to find a price, such a penalization is desired. A perfect hedge is better than a strategy which always outperform the claim, since the second one does not reflect the claim and therefore does not give an appropriate pricing information. Utility should be less. However, if we consider a pure investment problem, utility functions of this special form are not appropriate. That means if instead we have an increasing utility function, it does not play a role if we maximize among all superhedging or only among all hedging strategies. Both value functions have the same value. This will be interesting when we consider the connection between the exponential utility function and their even partial sums. The exponential utility function is increasing. Their even partial sums are not, hence from a certain initial wealth a superhedging strategy might be optimal. This problem vanishes when we choose the approximation better and better.

Including "real" Consumption in the Investment Problem

Note, up to know consumption was just an instrument to remove money from the portfolio to obtain a wealth process Y, which terminal wealth $Y_T =: f$ has (approximately) the same payoffs as the claim ξ. f might not be reachable by a usual hedging strategy. The fact "consumption" does not yield any increase in utility, only in the sense that we have a chance to better reach the claim ξ. Substitutional effects are not desirable when we like to find a price. However, in a pure investment or a mixed problem superhedging was not necessary to reach a claim. It even decreases utility, so consumption is only interesting when it is rewarded. In fact, it is quite unrealistic consumption would not increase our general satisfaction. Valuing consumption in the present against future consumption - investment - plays an important role in Economics. Therefore, we will widen our attention and at least state a more general model which includes consumption in our considerations. For simplicity, we assume that $C_t = \int_0^t c(s)ds$ for the next problem:

$$V(x)^C_\xi \equiv \sup_{Y \in \mathcal{W}_C(x)} E[\int_0^T U_1(t, c(t))dt + U_2(Y_T - \xi)], \; x \in \mathcal{E} \tag{2.5}$$

A solution for utility functions of type I is discussed in Karatzas (1996).

2.2 Reformulation of the Dynamic to a Static Problem

In the following chapters, we apply methods from convex analysis, therefore we have to include our problem into the framework of convex analysis. Optimization over a class of stochastic processes is difficult, we rather want to optimize over a class of random variables satisfying some constraints, but not dependent on time. This section is therefore devoted to the reformulation of the original dynamic problem. We consider all L^p-integrable $\mathcal{F}_T$-measurable random variables f satisfying the constraint: $E_Q(f) \leq x$ for all Q. If the class of trading strategies is chosen unfortunate, the set of all L^p-integrable $\mathcal{F}_T$-measurable random variables might be too big. In our model this is not the case. Delbaen and Schachermayer (1996b) prove a result that solves this problem for L^p-hedging strategies (Theorem 2.2.5). We generalize this to L^p-superhedging strategies see Theorem 2.2.10.

After successfully reformulating the dynamic problem to a static problem of convex analysis, in chapter 3 we pose the dual problem in the sense of convex analysis and solve it. Afterwards we can try to build a solution for the constraint static problem by using the solution of the dual problem. The constraint static problem is equivalent to the dynamic problem, hence we obtain a solution for our original problem. We know that the optimal random variable is a terminal value of a process generated by a strategy in $\mathcal{A}^p \times \mathcal{K}^p$. We do not obtain the strategy explicitly.

2.2.1 Complete Case

In a complete market every contingent claim is attainable by definition. A common characterization of this assertion is that the set of martingale measure is a singleton $\mathcal{M}_a = \{Q_0\}$. Recall, we assume that this equivalence is given in our models. Reformulating the dynamic problem into a static problem is therefore easier in the complete case. We treat this case in this subsection. More explicitly: An arbitrage-free market is complete, if every contingent claim f, i.e. a $\mathcal{F}_T$ measurable random variable with $E(f^p) < \infty$, $p \in (1, \infty)$, is attainable. Note, if $Z_{Q_0} \in L^q(P), \frac{1}{p} + \frac{1}{q} = 1$, we have by Hölder's inequality that $u_{Q_0} := E_{Q_0}(f) < \infty$. Attainable means that there exists an allowable portfolio such that such that for the generated wealth process Y holds: $Y_T^{u_0,\pi,C=0} = f$. So the contingent claim is attainable with an initial wealth of x if and only if $E_{Q_0}(f) = u_0 = x$. We define:

Definition 2.2.1. The seller price h_{up} and the buyer price h_{low} are given by::

$$h_{up}(f) = \inf H,\ H := \{x \geq 0 : \exists (N_x, C_x) \in \mathcal{A} \times \mathcal{K},\ Y_T^{C_x,x,N_x} \geq f \text{ a.s}\}$$
$$h_{low}(f) = \sup L\ L := \{x \geq 0 : \exists (N_x, C_x) \in \mathcal{A} \times \mathcal{K},\ Y_T^{C_x,-x,N_x} \geq -f \text{ a.s}\}$$

We have:

Lemma 2.2.1. *In the proposed semimartingale model with* $\mathcal{M}_e^q = \{Q_0\}$ *(market is complete), let* f *be an* $\mathcal{F}_T$*- measurable random variable with* $E(f^p) < \infty$*, then the following holds:*

1. *The existence of an* $Y^{x,N,0} \in \mathcal{W}(x)$ *such that* $Y_T^{x,N,C} = f$*, i.e.* f *is attainable, is equivalent to* $E_{Q_0}[f] = x$*.*
2. *The existence of an* $Y^{x,N,C} \in \mathcal{W}_C(x)$ *such that* $Y_T^{x,N,C} = f$*, i.e.* f *can be superhedged with* x*, is equivalent to* $E_{Q_0}[f] \leq x$

Proof. The first equivalence is obvious by definition of completeness. We know that every claim is attainable. Define $u_f = E_{Q_0}(f)$, so for every f there exists a portfolio process π such that $Y_T^{u_f,\pi,0} = f$. Now suppose f can be superhedged with x, since f can be hedged with u_f and $u_f = h_{up}$ is equal to the infimum of all initial wealth such that f can be superhedged, we have $u_f \leq x$ and therefore $E_{Q_0}(f) \leq x$. Next assume $E_{Q_0}[f] \leq x$, hence $u_f \leq x$. That means we hedge the claim with u_f and consume the rest, i.e. $C_T = x - u_f$. □

Note, u_f is element of H and L in the definition of h_{up} and h_{low} and we take infimum and supremum, it follows that $h_{up} = h_{low}$, therefore $u_f = h_{up}$. By Lemma 2.2.1 the problem (2.3) is equivalent to the following static problem:

$$V(x)_\xi \equiv \sup_{f \in L^p(\mathcal{F}_T), E_{Q_0}(f) \leq x} E[U(f - \xi)], \; x \in \mathcal{E} \tag{2.6}$$

2.2.2 Incomplete Case

In this section, we describe the reformulation in an incomplete market. We show that under certain assumption it is equivalent to optimize over the terminal values of a wealth process with initial wealth x or over L^p-random variables f satisfying a budget constraint $\sup_{Q \in \mathcal{M}_a^q} E_Q(f) \leq x$.

The reformulation of the problem in the incomplete case is more complicated, since not every claim is attainable. There exists more than one martingale measure. We want their densities to be in L^q. That is satisfied when optimizing over the set $\mathcal{G}^p(x)$.

However, we have to prove that this set is closed in L^p and convex. The convexity is clear. Closedness is proven in Grandits and Krawczyk (1998) under assumption 1.2.4. Further this class is very abstract, we need a good criterion to treat this class. We prove:

$$\mathcal{G}^p(x) = \{f \in L^p : \forall Q \in \mathcal{M}_e^q \; E_Q(f) = x\}. \tag{2.7}$$

for $Y \in \mathcal{W}(x)$. So problem (2.3) is equivalent to:

$$V(x)_\xi \equiv \sup_{f \in L^p(P) | \forall Q \in \mathcal{M}_e^q \; E_Q(f) = x} E[U(f - \xi)], \; x \in \mathcal{E}. \tag{2.8}$$

In Theorem 2.2.10, we further see that problem (2.3) is also equivalent for processes Y in $\mathcal{W}_C$, i.e. the class of all wealth process generated by a superhedging strategy:

$$V(x)_{\xi,C} \equiv \sup_{f\in L^p(P)|\forall Q\in\mathcal{M}_e^q\ E_Q(f)\leq x} E[U(f-\xi)],\ x\in\mathcal{E}. \tag{2.9}$$

Further, optimizing over $\mathcal{M}_e^q$ in the dual problem might lead to a maximum outside this set. So in some cases it is necessary to switch to $\mathcal{M}_S^q$. Fortunately, by Theorem 4.1.13 for special utility functions, (2.9) is equivalent to the following problem and the maximizer is in $\mathcal{M}_e^q$, see section 4.1.3:

$$V(x)_{\xi} \equiv \sup_{f\in L^p(P)|\forall Q\in\mathcal{M}_S^q\ E_Q(f)= x} E[U(f-\xi)],\ x\in\mathcal{E}. \tag{2.10}$$

We start showing (2.7) in a Brownian setting with bounded coefficients. We need some preparation, we modify some results from Kohlmann (2003), p.100/101:

Definition 2.2.2. A self-financing portfolio π is called a Föllmer-Schweizer-Portfolio (FS) against f, if there exists a square integrable martingale M_t with expectation zero and a $\mathcal{F}_t$ progessively measurable process $Y\in\mathcal{S}^{c,p}$ such that

$$dY_t = \pi'\mu dt + \pi'\sigma dW_t + dM_t, Y_T = f,$$

M_t has to be orthogonal to $\int_0^t \sigma_s dW_s$

Theorem 2.2.2. *For a contingent claim $f\in L^p(P)$, a FS- portfolio always exists. It is a function of the solution (Y,u) of the following BSDE:*

$$dY_t = u_t\bar{\theta}dt + u_t dW_t,\quad Y_T = f,\quad \bar{\theta} = \sigma'(\sigma\sigma')^{-1}\mu.$$

The portfolio and the square-integrable martingale is given by

$$\pi = (\sigma\sigma')^{-1}\sigma u'\quad M_t = \int_0^t \tilde{u} dW_s,\quad \text{where } \tilde{u}_t = u_t - \pi_t'\sigma_t.$$

Proof. See Kohlmann (2003) p.101 for $p=2$. Up to some assumptions concerning the measurability, the BSDE above has a solution in $p\in(1,2)$, see Briand et al (2003). So Theorem 2.2.2 holds for any $p\in(1,\infty)$ (for $p>2$ since $L^p(P)\subset L^2(P)$ since P is a probability measure.) □

Note, $\mathcal{E}(\int_0^T \bar{\theta}_s dW_s)$ coincides with the density of the minimal martingale measure, which is defined in section 4.1.1. By Theorem 2.2.2, we show that (constrained) f are attainable. We prove that $\tilde{u}=0$:

Lemma 2.2.3. *In the proposed Brownian Model, let f be $\mathcal{F}_T$- measurable L^p-integrable random variable. $\sup_{Q\in\mathcal{M}_e^q} E_Q(f)<\infty$ follows and the following holds:*

1. *The existence of an* $Y \in \mathcal{W}(x)$ *such that* $Y_T = f$*, i.e.* f *is attainable with* x*, is equivalent to* $\forall Q \in \mathcal{M}_e^q : \; E_Q[f] = x$*.*

2. *If there exists a* $Y \in \mathcal{W}_C(x)$ *such that* $Y_T = f$*, i.e.* f *can be superhedged with* x*, then* $\sup_{Q \in \mathcal{M}_e^q} E_Q[f] \leq x$ *and* $\forall Q \in \mathcal{M}_e^q : \; E_Q[f] \leq x$*, respectively.*

Moreover, $E_Q[f] = x$ *only has to be satisfied for all measures generated by bounded* θ*.*

Proof. We start with item 1. Suppose f is attainable with x then there exists a portfolio π such that $Y_T^{x,\pi,0} = f$. The corresponding wealth process is a martingale under all Q (see Lemma 1.2.5), hence for all Q, we have $E_Q(Y_T^{x,\pi,0}) = x$ and:

$$u_Q = E_Q(f) = E_Q(Y_T^{x,\pi,0}) = x \Rightarrow \forall Q, \tilde{Q} \in \mathcal{M}_e^q : u_Q = u_{\tilde{Q}}$$

Analogously for the second item: Suppose f is superhedgable with x then there exists a portfolio (π, C) such that $Y_T^{x,\pi,C} = f$. The corresponding wealth process is a Q-supermartingale under all Q, hence for all Q, we have $E_Q(Y_T^{x,\pi,C}) \leq x$. Thus: $u_Q = E_Q(f) = E_Q(Y_T^{x,\pi,C}) \leq x$.

Next, we show the second part of the first item: Assume $\forall Q : \; E_Q[f] = x$. By Theorem 2.2.2, we have to show that $\tilde{u} = 0$. Firstly, consider the expectation of M_T under the minimal martingale measure $E^{\bar{\theta}} := E_{Q_{mmm}}$:

$$\begin{aligned} E^{\bar{\theta}}(M_T) &= E^{\bar{\theta}}\Big(\int_0^T \tilde{u}_s dW_s\Big) \\ &= E^{\bar{\theta}}\Big(\int_0^T \tilde{u}_s dW_s^{\bar{\theta}} + \int_0^T u_s \bar{\theta}_s ds\Big) \\ &= E^{\bar{\theta}}\Big(\int_0^T \tilde{u}_s \bar{\theta}_s ds\Big) = 0 \end{aligned} \tag{2.11}$$

since $\tilde{u}_s \in ker(\sigma)$ and $\bar{\theta}_s \in im(\sigma') = ker(\sigma)^\perp$ are orthogonal. $E_Q(f) = E_Q(f - M_T) + E_Q(M_T)$ and $f - M_T$ is attainable, so its expectation under every measure is equal. So by assumption $E_Q(M_T)$ takes the same value under every measure which is zero by (2.11). Define

$$\theta_s^N = \bar{\theta}_s + 1_{(|\tilde{u}_s| \leq N)} \tilde{u}_s =: \bar{\theta}_s + \bar{\theta}_s^N.$$

We know $\tilde{u}_s \in ker(\sigma_s)$, therefore also $\bar{\theta}_s^N \in ker(\sigma_s)$. A solution plus an element of the kernel is again a solution of a linear system of equations, hence:

$$\sigma_s \bar{\theta}_s^N = \mu,$$

and

$$\begin{aligned} 0 &= E^{\bar{\theta}^N}(M_T) \\ &= E^{\bar{\theta}^N}(\int_0^T u_s \bar{\theta}_s^N ds) \\ &= E^{\bar{\theta}^N}(\int_0^T 1_{(|\tilde{u_s}| \leq N)} |\tilde{u}_s|^2 ds) \\ &\geq \frac{1}{N^2} Q^{\bar{\theta}_N} \otimes \lambda_{[0,T]}(\{\frac{1}{N} \leq |\tilde{u}| \leq N\}) \\ &\Rightarrow \ \forall N \in \mathbb{N}: \ P \otimes \lambda_{[0,T]}(\{\frac{1}{N} \leq |\tilde{u}| \leq N\}) = 0 \\ &\Rightarrow \ P \otimes \lambda_{[0,T]}(\{\tilde{u} \neq 0\}) = 0 \Rightarrow \ \tilde{u} = 0 \quad P \otimes \lambda_{[0,T]} - a.s \end{aligned}$$

The first implication follows since $Q_{\bar{\theta}^N} \sim P$ and note, $1_{(|\tilde{u_s}| \leq N)} |\tilde{u}_s|^2 \in L^1(Q^{\bar{\theta}})$. We have only used bounded θ. $\bar{\theta}$ is bounded since μ is bounded and $\exists e : (\sigma\sigma')^{-1} \leq e$ by assumption 1.2.5. $\tilde{\theta}_s^N$ are bounded by definition. □

Lemma 2.2.3 holds in a more general setting. We use Theorem 1.2 in Delbaen and Schachermayer (1996b) for the case of a continuous process (see also Theorem 2.2. and the discussion above of it and note that a continuous S already implies that S_t is locally in L^p):

Theorem 2.2.4. *Let $p \in [1, \infty]$, q its conjugate exponent, S a semimartingale locally in $L^p(P)$ such that $\mathcal{M}_e^q \neq \emptyset$, and $f \in L^p(P)$. The following assertions are equivalent:*

1. $f \in \overline{\mathrm{V}^p(0)} = \overline{\mathcal{G}^p(0)}$
2. *There is an $S-$integrable predictable process N such that, for each $Q \in \mathcal{M}_e^q$ the process $(\int^t N_u dS_u)_t$ is a uniformly integrable $Q-$martingale converging to f in the norm of $L^1(Q)$.*
3. *There is an $S-$integrable predictable process N such that, for each $Q \in \mathcal{M}_a^q$ the process $(\int^t N_u dS_u)_t$ is a uniformly integrable $Q-$martingale converging to f in the norm of $L^1(Q)$.*
4. *There is an $S-$integrable predictable process N such that, for some $Q \in \mathcal{M}_e^q$ the process $(\int^t N_u dS_u)_t$ is a uniformly integrable $Q-$martingale converging to f in the norm of $L^1(Q)$.*
5. $E_Q(f) = 0$ *for each* $Q \in \mathcal{M}_e^q$
6. $E_Q(f) = 0$ *for each* $Q \in \mathcal{M}_a^q$

Note, $\overline{\mathcal{G}^p(x)} = \mathcal{G}^p(x)$, if condition 1.2.4 is satisfied.

Lemma 2.2.5. *In the proposed Semimartingale Model, $\mathcal{M}_e^q \neq \emptyset$, $f \in L^p$ be an $\mathcal{F}_T$- measurable random variable and assmuption 1.2.4 be satisfied, then the following holds:*

$$f \in \mathcal{G}^p(x) \Leftrightarrow f \in \overline{\mathrm{V}^p(x)} \Leftrightarrow \forall Q \in \mathcal{M}_e^q: \; E_Q(f) = x \tag{2.12}$$

So

1. *The existence of an $Y \in \mathcal{W}(x)$ such that $Y_T = f$, i.e. f is attainable with x, is equivalent to $\forall Q \in \mathcal{M}_e^q: \; E_Q[f] = x$.*
2. *If there exists a $Y \in \mathcal{W}_C(x)$ such that $Y_T = f$, i.e. f can be super-hedged with x, then it follows that $\sup_{Q \in \mathcal{M}_e^q} E_Q[f] \leq x$, $\forall Q \in \mathcal{M}_e^q: E_Q[f] \leq x$, respectively.*

Proof. The last $\Leftrightarrow$ follows from Theorem 2.1 in Delbaen and Schachermayer (1996b). The first equivalence holds since under assumption 1.2.4 $\mathcal{G}^p$ is closed by Theorem 1.2.4 and by Lemma 1.2.3. The second assertion follows as in the proof of Lemma 2.2.3. □

We would like to have also the reverse implication of the last item. However, if we have an increasing utility function the first part is enough the supremum over all f which satisfy the equality is the same if you also allow for f which satisfy the inequality. In the Brownian case we further define:

Definition 2.2.3. $\mathcal{Q}_.^{\,b}$ is the class of all martingale measures with density:

$$Z = \exp\left\{ -\int_0^T \theta_s' dW_s - \frac{1}{2} \int_0^T \|\theta_s\|^2 ds \right\}$$

where θ is a bounded solution of $\sigma\theta = \mu$.

By Lemma 2.2.3 and Theorem 1.2.5, problem (2.3) for increasing utility functions in the Brownian Model is equivalent to:

$$V(x)_{\xi,i} \equiv \sup_{f \in L_p(\mathcal{F}_T), \forall \theta \in Q_.^{\,b} E_{Q^\theta}(f) \leq x} E[U(f - \xi)], \; x \in \mathcal{E} \tag{2.13}$$

and in the general case

$$V(x)_{\xi,i} \equiv \sup_{f \in L_p(\mathcal{F}_T), \forall Q \in \mathcal{M}_e^q E_Q(f) \leq x} E[U(f - \xi)], \; x \in \mathcal{E} \tag{2.14}$$

For not necessary increasing functions, we only have equalities and $i \neq C$.

Note, in the first case, we restricted the class of martingale measures - we only take equivalent martingale measures with bounded θ. To solve the problem (2.14) we will switch to a dual problem over these measures. The restricted class is small, so it is more difficult to show that the supremum of the dual problem is attained. However, in the Brownian case with an exponential utility function (see Rouge and El Karoui (2000)), optimizing over bounded θ leads to the same minimizer.

If we are in a mean variance hedging problem, we also have an increasing function up to a certain point. In the other part, the superhedging strategy is quite obvious by hedging the maximum and consuming the rest. When using Lemma 2.2.3, we just have

$$\sup_{f:\forall Q E_Q(f)\leq x} E[U(f-\xi)] \geq \sup_{(\pi,C)\in\mathcal{A}\times\mathcal{K}} E[U(Y_T^{x,\pi,C}-\xi)], \tag{2.15}$$

because of the last implication of Lemma 2.2.3. So in the end we have to check, if the optimal solution of the static problem is also one of the dynamic problem.

Lemma 2.2.3 is discussed in Karatzas (1996) for the Brownian case or more generally in Jacka (1992), Delbaen and Schachermayer (1994), Delbaen and Schachermayer (1995b), El Karoui and Quenez (1995), Ansel and Stricker (1994). However, they either use different classes of strategies or claims have to be non-negative. Fortunately, it turns out that the corresponding proofs do not depend on this very heavily and we can modify them to establish the reverse implication in Lemma 2.2.5. We closely follow the approach in El Karoui and Quenez (1995). Lemma 2.2.5. is not explicitly stated there, furthermore its underlying setting is a Brownian model. So we need some preparation to transform the proof into the general semimartingale setting. We follow the approach of Schweizer (1995): A P-semimartingale S satisfies the structure condition (SC) if

1. S is of the form

$$S = S_0 + M + A,\ M \in \mathcal{M}^2_{0,loc}(P), \tag{2.16}$$

where $A^{(i)} \ll \langle M^{(i)}\rangle$, $A^{(i)}$ with predictable density $\alpha^{(i)}$, $(\int \alpha^{(i)} d\langle M^{(i)}\rangle)$ for $i \leq n$.

2. There exists a predictable process $\hat{\lambda} \in L^2_{loc}(M)$ with

$$d\langle M\rangle_t \hat{\lambda}_t = \gamma_t \text{ P-a.s. for } t \in [0,T] \tag{2.17}$$

where $d\langle M\rangle$ is a $n\times n-$matrix with components $d\langle M^{(i)}, M^{(j)}\rangle_t$ and γ_t is a $n \times 1$- vector with components $\gamma_t^{(i)} = \alpha_t^{(i)} d\langle M^{(i)}\rangle_t$. Note, $\int \hat{\lambda}' dM$ does not depend on the choice of $\hat{\lambda}$, see Jacod (1979). In the Brownian model, it is obvious because only one solution $\hat{\lambda}$ exists, provided $\sigma\sigma'$ is positive definite.

If the structure condition is satisfied for S, then

$$\hat{Z} = \mathcal{E}(-\int \hat{\lambda}' dM) \tag{2.18}$$

is a martingale density process, i.e. a real-valued, local $P-$martingale with $Z_0 = 1$ P-a.s. such that the product SZ is a local $P-$martingale. We always use the RCLL-modification of it. Z is called strict if it is strictly positive. It is a strict martingale density process if and only if $\hat{\lambda}'_t \Delta M_t < 1$ P-a.s. for $t \in [0, T]$, this is automatically satisfied in our case since $S = S_0 + M + A$ is continuous. A continuous process admitting a strict martingale density process satisfies the structural condition and vice versa (Theorem 1 and Proposition 2 Schweizer (1995)). If $\hat{Z}$ is a martingale, Schweizer (1995) (p.7 second definition) call the martingale measure induced by $\hat{Z}$ the minimal martingale measure:

$$\frac{d\hat{Q}}{dP} = \hat{Z}_T, \quad \hat{Z}_T = \mathcal{E}_T(-\int \hat{\lambda}' dM)$$

In section 4.1.1, we will see that under the condition that $\hat{Z}_T$ is square-integrable, this corresponds to a martingale measure Q_{mmm} that preserves the structure of P best (see below definition 4.1.6 extending definition 4.1.2. Since for most applications, it is assumed that $\langle -\int \hat{\lambda}' dM \rangle_T$ is bounded, the equivalence is given. So we later set $\hat{Q} = Q_{mmm}$. Furthermore, we establish in the same section that structure condition is satisfied in our Brownian setting and that $\hat{Z}_T$ corresponds to $\mathcal{E}_T(\int \bar{\theta}_s dW_s)$. In the general case, we cite a theorem from Schweizer (1995) (Theorem 1) to indicate when the structure condition is satisfied in a more general setting:

Theorem 2.2.6. *Suppose there exists a strict martingale density process Z for S (i.e. a $P-$martingale with $Z_0 = 1$, such that SZ is a local $P-$martingale) and S is continuous then S satisfies the structure condition and $\alpha^{(i)} \in L^2_{loc}(M^{(i)})$, $i \leq n$. Further, we have:*

$$Z = \mathcal{E}(-\int \hat{\lambda}' dM + L) = \mathcal{E}(-\int \hat{\lambda}' dM)\mathcal{E}(L) \tag{2.19}$$

where $L \in \mathbb{M}_{0,loc}(P)$ is strongly orthogonal to $M^{(i)}$ for every i. If S is possibly discontinuous, but instead satisfies (2.16) and $Z \in \mathbb{M}^2_{loc}(P)$, we obtain the same result, but we only have:

$$Z = \mathcal{E}(-\int \hat{\lambda}' dM + L), \tag{2.20}$$

where $L \in \mathbb{M}^2_{0,loc}(P)$ is strongly orthogonal to $M^{(i)}$ for every i.

The second equality in equation (2.19) follows since S is continuous. (Orthogonality therefore means that $\langle \int \hat{\lambda}' dM, L \rangle = 0$). The assertion follows. The Theorem is a generalization of results in Ansel and Stricker (1992) and Ansel and Stricker (1993a). It relies on the Kunita-Watanabe-decomposition. According to Lemma 1.2 in Ansel and Stricker (1993a) for any strict martingale density process there exists a local martingale K such that $Z = \mathcal{E}(K)$. If S is continuous then M of $S = S_0 + M + A$ is also continuous. By Ansel and Stricker (1993b) (third paragraph, case 3), K can be decomposed in $K = \int HdM + L$, where L is a local martingale orthogonal to M^i for every $i = 1, ..., n$ and H predictable.

If S is continuous, then all martingale densities are continuous as well and therefore also L in Theorem 2.2.6. See Jacod (1979) for further details. Since continuity is our standard assumption, orthogonality just means that the cross-variation $\langle N, M\rangle$ is zero.

Definition 2.2.4. 1. $\mathcal{N}$ is the class of all local martingales N with $N_0 = 0$ orthogonal to M such that $\mathcal{E}_t(-\int \hat{\lambda}' dM + N)$, $t \in [0,T]$ is a martingale under P

2. $\mathcal{N}_q$ is a subclass of $\mathcal{N}$, where $\mathcal{E}_t(-\int \hat{\lambda}' dM + N)$, $t \in [0,T]$ is a strictly positive $P-$martingale with $E\mathcal{E}_t^q(-\int \hat{\lambda}' dM + N) < \infty$, $t \in [0,T]$ in addition.

By Theorem 2.2.6 in the continuous case and as in El Karoui and Quenez (1995), we define the following one-to-one mapping:

$$N \mapsto Q^N,$$

where

$$dQ^N = \mathcal{E}_t^q(-\int \hat{\lambda}' dM + N)dP = \mathcal{E}_T(N)\mathcal{E}_t^q(-\int \hat{\lambda}' dM)dP = \mathcal{E}_T(N)dQ_{mmm}$$

by Theorem 2.2.6. Q_{mmm} denotes the minimal martingale measure. Hence,

$$\begin{aligned} \mathcal{M}_e &= \{Q : Q \sim P : \frac{dQ}{dP} = \mathcal{E}_T(-\int \hat{\lambda}' dM + N),\ N \in \mathcal{N}\} \\ \mathcal{M}_e^q &= \{Q : Q \sim P : \frac{dQ}{dP} = \mathcal{E}_T(-\int \hat{\lambda}' dM + N),\ N \in \mathcal{N}_q\} \end{aligned}$$

Finally,

$$\operatorname{ess}\sup_{Q \in \mathcal{M}_e} E_Q(f|\mathcal{F}_t) = \operatorname{ess}\sup_{N \in \mathcal{N}^q} E_{Q^N}(f|\mathcal{F}_t) \tag{2.21}$$

Using this reformulation, we extend a result in El Karoui and Quenez (1995) (Theorem 2.1.1.):

Theorem 2.2.7. *There exists an RCLL process $(J_t, 0 \le t \le T)$ with*

$$\forall t \in [0,T]:\ J_t = \operatorname{ess}\sup_{N \in \mathcal{N}^q} E_{Q^N}(f|\mathcal{F}_t).$$

J_t is the smallest right continuous supermartingale under Q_N with $J_T = f$, for every $N \in \mathcal{N}_q$. N^{opt} is optimal, i.e. $J_t = E_{Q_{N^{opt}}}(f|\mathcal{F}_t)$ $P-a.s.$ if and only if J_t is a martingale under $Q_{N^{opt}}$.

The proof is given later. We want to establish that J_t is the desired wealth process. J_t is a RCLL-supermartingale. We cite a theorem from Föllmer and Kabanov (1998) adjusted to our setting:

Theorem 2.2.8. *Suppose $\mathcal{M}_e \neq \emptyset$. Let J be a right-continuous local semimartingale with respect to any Q^N with $N \in \mathcal{N}_q$. Then there exist an increasing right-continuous adapted process C with $C_0 = 0$ and a predictable integrand N such that $J = J_0 + \int NdS - C$.*

We have $J_T = f$. It remains to show that N and C are sufficiently integrable to be a L^p-trading strategy. The next Lemma gives the necessary integrability of J :

Lemma 2.2.9. *J as defined in Theorem 2.2.7 satisfies:*

$$\|J_0\|_{L^p(P)} + \|\langle \int NdM\rangle_T^{\frac{1}{2}} + \int_0^T |NdA_s| + C_T\|_{L^p(P)} < \infty \tag{2.22}$$

A proof, using that $f \in L^p(P)$, is given at the end of this section. Since all summands in (2.22) are non-negative this yields:

$$\|\langle \int NdM\rangle_T^{\frac{1}{2}}\|_{L^p(P)} + \|\int_0^T |NdA_s|\|_{L^p(P)} < \infty \Rightarrow N \in \mathcal{A}^p$$

and

$$\|C_T\|_{L^p(P)} < \infty \Rightarrow C \in \mathcal{K}^p.$$

Since $J_0 = \sup_{Q \in \mathcal{M}_e} E_Q(f) \leq x$, $(x, N, C) \in \mathbb{R} \times \mathcal{A}^p \times \mathcal{K}^p$. Thus, (x, N, C) is a superhedging strategy according to definition 2.1.1. We summarize:

Theorem 2.2.10. *In the proposed semimartingale model, $\mathcal{M}_e^q \neq \emptyset$, $f \in L^p(P)$ be $\mathcal{F}_T$- measurable random variable, then $\sup_{Q \in \mathcal{M}_e^q} E_Q(f) < \infty$. If in addition, assumption 1.2.4 is satisfied, the following holds:*

$$f \in \mathcal{G}^p(x) \Leftrightarrow f \in \overline{V^p(x)} \Leftrightarrow \forall Q \in \mathcal{M}_e^q : \ E_Q(f) = x \tag{2.23}$$

So

1. *The existence of a $Y \in \mathcal{W}(x)$ such that $Y_T = f$, i.e. f is attainable with x, is equivalent to $\forall Q \in \mathcal{M}_e^q : \ E_Q[f] = x$.*
2. *The existence of a $Y \in \mathcal{W}_C(x)$ such that $Y_T = f$, i.e. f can be superhedged with x, is equivalent to $\sup_{Q \in \mathcal{M}_e^q} E_Q[f] \leq x$.*

So

$$V(x)_{\xi,C} \equiv \sup_{Y \in \mathcal{W}_C(x)} E[U(Y_T - \xi)], \ x \in \mathcal{E}$$

is equivalent to

$$V(x)_{\xi,C} \equiv \sup_{f \in L_p(\mathcal{F}_T), \forall Q \in \mathcal{M}_e^q E_Q(f) \leq x} E[U(f - \xi)], \ x \in \mathcal{E} \tag{2.24}$$

for utility functions U with $E(U(X)) < \infty, \ X \in L^p$. We now give a proof of Theorem 2.2.7, mainly taken from El Karoui and Quenez (1995):

Proof. (Proof of Theorem 2.2.7)
The proof is taken from El Karoui and Quenez (1995). First remark that J_t is only defined almost surely. However, we know:

$$\forall N \in \mathcal{N}_q \ \forall t \in [0,T]: \ E_{Q^N}(f|\mathcal{F}_t) = E_{Q^{\tilde{N}}} E(f|\mathcal{F}_t) \tag{2.25}$$

where $\tilde{N}_u = N_u - N_{t \wedge u}, \ 0 \leq u \leq T$. Thus,

$$J_t = \text{ess} \sup_{N \in \mathcal{N}_q(t)} E_{Q^N}(f|\mathcal{F}_t), \ \mathcal{N}_q(t) = \{N \in \mathcal{N}_q(t) | N_u = 0 \ \forall u \in [0,t]\} \tag{2.26}$$

is a properly defined process. For a $N \in \mathcal{N}_q)(t)$, we set:

$$\Gamma(t,N) = E_{Q^N} E(f|\mathcal{F}_t) = E_{Q_{mmm}}[\mathcal{E}(N)_T f|\mathcal{F}_t] \tag{2.27}$$

The class $\{\Gamma(t,N), N \in \mathcal{N}_q(t)\}$ is stable by supremum and infimum. Hence, for each t there exists a sequence $N_p \in \mathcal{N}_q(t)$ such that almost surely, $\Gamma(t,N_p)$ is an increasing sequence of random variables converging to J_t, i.e.:

$$J_t = \lim_{p \to +\infty} \uparrow \Gamma(t,N_p) = \lim_{p \to +\infty} \uparrow E_{Q^{N_p}}(f|\mathcal{F}_t) \tag{2.28}$$

We still have to prove the above stability. It suffices to show that:

$$N_1, N_2 \in \mathcal{N}_q(t) \Rightarrow \exists N \mathcal{N}_q(t) : \Gamma(t,N) = \Gamma(t,N_1) \vee \Gamma(t,N_2)$$

We further set $A = \{\Gamma(t,N_1) \leq \Gamma(t,N_2)\}$ and $N = 1_{A^c}\Gamma(t,N_1) + 1_A \Gamma(t,N_2)$. We have $A \in \mathcal{F}_t$, $N \in \mathcal{N}_q(t)$ and

$$\begin{aligned} \Gamma(t,N) &= E_{Q_{mmm}}(\mathcal{E}(N_1)_T f|\mathcal{F}_t) 1_{A^c} + E_{Q_{mmm}}(\mathcal{E}(N_2)_T f|\mathcal{F}_t) 1_A \\ &= \Gamma(t,N_1) 1_{A^c} + \Gamma(t,N_2) 1_A = \Gamma(t,N_1) \vee \Gamma(t,N_2) \end{aligned}$$

Analogously for the infimum. Next, take a sequence as in (2.28), such that $J_t = \lim_{p \to +\infty} \uparrow \Gamma(t,N_p)$. Then, for any $N \in \mathcal{N}_q$, J_t is a supermartingale under Q_N, i.e. $\mathcal{E}(N)_T J_t$ is a supermartingale under the minimal martingale measure Q_{mmm} : Since,

$$\begin{aligned} E\left[\frac{\mathcal{E}(N)_t}{\mathcal{E}(N)_s} J_t | \mathcal{F}_s\right] &= \lim_{p \to +\infty} \uparrow E\left[\frac{\mathcal{E}(N)_t}{\mathcal{E}(N)_s} E(\mathcal{E}(N_p)_T f|F_t)|\mathcal{F}_s\right] \\ &= \lim_{p \to +\infty} \uparrow E\left[\frac{\mathcal{E}(N)_t}{\mathcal{E}(N)_s} \mathcal{E}(N_p)_T f|\mathcal{F}_s\right] \\ &= \lim_{p \to +\infty} \uparrow \left[\frac{\mathcal{E}(\tilde{N}_p)_t}{\mathcal{E}(\tilde{N}_p)_s} f|\mathcal{F}_s\right] \end{aligned}$$

where $\tilde{N}_p(u) = N(u \wedge t) + N_p(u), \ u \in [0,T]$ and therefore $(\tilde{N}_p(u), \ u \in [0,T]) \in \mathcal{N}_q$. We have,

$$E\left[\frac{\mathcal{E}(N)_t}{\mathcal{E}(N)_s} J_t | \mathcal{F}_s\right] \leq J_s$$

The remaining assertions of this Theorem are not necessary to prove Theorem 2.2.10, so we refer to El Karoui and Quenez (1995) for the rest of the proof. □

Finally, we prove Lemma 2.2.9:

Proof. (Proof of Lemma 2.2.9) Define $m_t^{Q_n} = E_{Q_n}(f|\mathcal{F}_t)$. Further, by Theorem 2.2.8, we know that J_t is of the form:

$$J_t = J_0 + m_t - a_t,$$

where m is a continuous martingale and a an increasing process of finite variation. $J_0 = \sup_{Q \in \mathcal{M}_e^q} E_Q(f) < \infty$ ($f \in L^p(P)$) and so also $\|J_0\|_{L^p(P)} < \infty$. Since, we are only interested in $\|\langle \int NdM\rangle_T^{\frac{1}{2}} + \int_0^T |NdA_s|\|_{L^p(P)} < \infty$, it suffices to consider $J_0 = 0$. We show that:

1. $\langle m^{Q_n}\rangle_T^{\frac{1}{2}} \overset{P-a.s}{\longrightarrow} (\langle m\rangle_T + \int_0^T |da_s^2|)^{\frac{1}{2}}$

2. $(\langle m^{Q_n}\rangle_T^{\frac{1}{2}})_n$ is p-times equi-integrable.

According to Theorem 20.4. in Bauer (1968) this yields that $\langle m^{Q_n}\rangle_T^{\frac{1}{2}}$ converges to $(\langle m\rangle_T + \int_0^T |da_s^2|)^{\frac{1}{2}}$ in $L^p(P)$.

$$\begin{aligned}
&\|\langle m\rangle_T^{\frac{1}{2}} + \int_0^T |da_s|\|_{L^p(P)} \\
= &\left\| \sqrt{\left(\langle m\rangle_T^{\frac{1}{2}} + \int_0^T |da_s|\right)^2} \right\|_{L^p(P)} \\
= &\left\| \sqrt{\langle m\rangle_T + 2\langle m\rangle_T^{\frac{1}{2}} \int_0^T |da_s| + \int_0^T |da_s^2|} \right\|_{L^p(P)} \\
\leq &\left\| \left(\langle m\rangle_T + 2 \cdot 2(\langle m\rangle_T + \int_0^T |da_s^2|) + \int_0^T |da_s^2|\right)^{\frac{1}{2}} \right\|_{L^p(P)} \\
= &\sqrt{5} \left\| \left(\langle m\rangle_T + \int_0^T |da_s^2|\right)^{\frac{1}{2}} \right\|_{L^p(P)} < \infty
\end{aligned}$$

the assertion follows.

We start showing the first item: We know from the last proof that:

$$\mathrm{G} := \{\langle m^{Q_n}\rangle_T | n \in \mathcal{N}^q\}$$

is closed under pairwise maximization, so as in the last Theorem (see also Theorem A.3 in Karatzas und Shreve (1998)), there exists a sequence $(Q_n)_n$ such that $m_t^{Q_n}$ converges J_t Q_n-almost surely for every Q_n. Since Q_n is equivalent to P convergence is also P-almost surely for n tending to infinity:

$$m_t^{Q_n} \to J_t \;\; P - a.s. \tag{2.29}$$

This also holds for the supremum, since $m_t^{Q_n}$ is continuous. So this together with a slight modification of item 2 already yield convergence in $S^p(P)$. However, we will not need this fact. m_t is real-valued and continuous, so we further know from Jacod (1979) p.171/172 that for any partition $0 \leq t_0 \leq t_1 \leq ... \leq t_l = T$, if we choose the partition finer and finer ("$l \to \infty$"):

$$\sum_{i=0}^{l} (m_{t_{i+1}}^{Q_n} - m_{t_i}^{Q_n})^2 \stackrel{P-a.s.}{\longrightarrow} \langle m^{Q_n} \rangle_T. \tag{2.30}$$

Note, this even holds for semimartingales. We introduce the stopping time R_k:

$$R_k = \inf\{t \geq 0 : \; \langle m \rangle_t > k, \; \forall n \in \mathbb{N} \; m_t^{Q_n} > k\}, \; k \in \mathbb{N}$$

Then the function

$$g_l(m_{t_i}^{Q_n}, \; i = 1, ..., l) = \sum_{i=0}^{l} (m_{t_{i+1}}^{Q_n} - m_{t_i}^{Q_n})^2$$

is uniformly continuous on $[0, R_k]$. Hence from (2.29), we have:

$$\begin{aligned} g_l(m_{t_i \wedge R_k}^{Q_n}) \quad & \stackrel{P-a.s.}{\longrightarrow} \quad \sum_{i=0}^{l} (m_{t_{i+1} \wedge R_k} - a_{t_{i+1} \wedge R_k} - m_{t_i \wedge R_k} + a_{t_i \wedge R_k})^2 \\ & = \quad \sum_{i=0}^{l} (m_{t_{i+1} \wedge R_k} - m_{t_i \wedge R_k})^2 + \sum_{i=0}^{l} (a_{t_{i+1} \wedge R_k} - a_{t_i \wedge R_k})^2 \\ & \qquad -2 \sum_{i=0}^{l} (m_{t_{i+1} \wedge R_k} - m_{t_i \wedge R_k})(a_{t_{i+1} \wedge R_k} - a_{t_i \wedge R_k}) \end{aligned}$$

Since a is a process of finite variation, the last part converges to $S(\Delta m \Delta a)$ the sum of the product of the jumps of m and a, see again Jacod (1979) page 171. m is continuous and therefore $S(\Delta m \Delta a)$ is zero.

We consider the diagonal sequence of

$$g_{n,k} := g_k(m_{t_i \wedge R_k}^{Q_n}) = \sum_{i=0}^{k} (m_{t_{i+1}}^{Q_n} - m_{t_i}^{Q_n})^2.$$

We set $k = l$ (or more generally $l = l(k) \to \infty$ if $k \to \infty$). By (2.30) we have for $k \to \infty$ and $n_k \to \infty$:

$$g(m_{t_i \wedge R_k}^{Q_{n_k}}) \quad \stackrel{P-a.s.}{\longrightarrow} \quad \langle m \rangle_T + \int_0^T da_s^2$$

Note, a is increasing since J is a supermartingale.

For the second item we use that $f \in \L^p(P)$. We have to show that:

$$\sup_{g \in \mathrm{G}} \int_\Omega |g|^p dP < \infty$$

and $\forall \epsilon > 0 \; \exists h \in L^p(P) \; h \geq 0 \; \exists \delta > 0 \; \forall A \in \mathcal{F}_T \; \forall g \in \mathrm{G}$:

$$\int_A h^p dP \leq \delta \Rightarrow \int_A |g|^p dP \leq \epsilon$$

According to Proposition 3.26 and Remark 3.27 in Karatzas und Shreve (1991) (Martingale Moment Inequality), we have:

$$E(\langle M \rangle_T^{\frac{p}{2}}) \leq C(p) E(|M_T|^p), \;\; \text{for } p > 1, \; M \in \mathbb{M}_{loc}^c$$

Further, we have by Bayes' rule for all $s \leq t \leq T$:

$$m_s^{Q_n} = E_{Q_n}(f|\mathcal{F}_s) = E(\frac{1}{Z_s} Z_t f|\mathcal{F}_s)$$

Hence, for $(M_s)_s = (\frac{1}{Z_s} Z_t f)_s$ and $s = t = T$:

$$E(\langle m^{Q_n} \rangle_T^{\frac{p}{2}}) \;\; \leq \;\; C(p) E(\frac{1}{Z_T} Z_T |f|^p||\mathcal{F}_T) = C(p) E(|f|^p)$$

Further, for a $A \in \mathcal{F}_T$ and since $(\langle m^{Q_n} \rangle_t)_t$ is $\mathcal{F}_t$-adapted, we have:

$$\begin{aligned} \int_A \langle m^{Q_n} \rangle_T^{\frac{p}{2}} dP &= E(\langle 1_A m^{Q_n} \rangle_T^{\frac{p}{2}}) \\ &\leq C(p) E(\frac{1}{Z_T} Z_T |1_A f|^p||\mathcal{F}_T) = C(p) \int_A |f|^p dP \end{aligned}$$

This yields:

$$\begin{aligned} \sup_{g \in \mathrm{G}} \int_\Omega |g|^p dP &= \sup_{Q_n} E(\langle m^{Q_n} \rangle_T^{\frac{p}{2}}) \\ &\leq C(p) E(|f|^p) < \infty \end{aligned}$$

and we set $h = |f|$ and have for every $\epsilon > 0$ an $\delta = \frac{\epsilon}{C(p)}$ such that for all $g \in \mathrm{G}$ and all $A \in \mathcal{F}_T$:

$$\sup_{g \in \mathrm{G}} \int_A |g|^p dP \leq C(p) \int_A |f|^p dP = C(p)\delta = \epsilon$$

□

We summarize:

Optional decomposition — Supermartingale — p-Integrability

Th. 2.2.8 ↘ Th. 2.2.7 ↓ ↙ L. 2.2.9

Theorem 2.2.10:

$$\sup_{Y \in \mathcal{W}_C(x)} E[U(Y_T - \xi)] \Leftrightarrow \sup_{f \in \mathcal{G}_C^p(x)} E[U(f - \xi)]$$

where $\mathcal{G}_C^p(x) = \{f \in L_p(\mathcal{F}_T) : \forall Q \in \mathcal{M}_e^q \ \ E_Q(f) \leq x\}$.

Chapter 3

Solution Procedure of the Static Problem

In this chapter, we apply results from convex analysis to obtain a solution for the static problem (2.24). We set $\xi \equiv 0$ and refer to section 3.2.3 for some ideas for an appropriate reformulation to the case of $\xi \neq 0$. We use that $\mathcal{G}^p \subset L^p$ and $\mathcal{M}^q_{Z,S} \subset L^q$ with p conjugate to q. L^p and L^q are dual to each other. These are rather strong assumptions. The approach in Kramkov and Schachermayer (1999) is more general: They do not impose any integrability conditions, they work with L^0. However, it is not ensured that the q-optimal measure exists, which we discuss in section 4.1.3. Further, the proofs in the general setting in Kramkov and Schachermayer (1999) are very technical, so the original idea might get lost. We treat these general aspects in section 3.2.2 (Utility Functions of Type II). Before we go on, we give a remark to the used notation:

Remark 3.0.1. (Notation) In a complete market, the set of martingale measures is a singleton. This unique measure is denoted by Q_0. A density process of Q_0 is not needed in this chapter. So the density of Q_0 is denoted by Z_{Q_0} instead of $Z_T^{Q_0}$. Similar in the incomplete case, we use Z as a shorthand for Z_T. If Z means the whole processes this is explicitly marked in this chapter.

3.1 A Second Reformulation Using Convex Analysis

The static problem is a convex optimization problem under constraints. In the sequel, we derive the Lagrange-functional of the static problem and show that it suffices to find a saddle point of this functional to obtain a solution of the original problem.

3.1.1 The Idea: Complete Case

We start with the complete case, since everything goes through more easily. Moreover, in some cases we can proceed similar in the incomplete case, where we face more than one equivalent martingale measure. Laxly formulated, we choose the measures, which leads to the worst market completion and use the result from the complete market. We have for all $\tilde{Q} \in \mathcal{M}_e^q$:

$$\begin{aligned} V(x) &:= \sup_{f \in L_p(\mathcal{F}_T), \forall Q \in \mathcal{M}_e^q E_Q(f) \leq x} E[U(f)],\ x \in \mathcal{E} \\ &\leq \sup_{f \in L_p(\mathcal{F}_T), \tilde{Q} \in \mathcal{M}_e^q E_{\tilde{Q}}(f) \leq x} E[U(f)],\ x \in \mathcal{E} \end{aligned}$$

The worst market completion means the value function or the corresponding measure, respectively, which comes closest or is equal to $V(x)$, i.e. the measure with the lowest value function. Note this might depend on the initial wealth, see Karatzas (1996)!

Recall, the device to solve the static problem is a further reformulation of the static problem to a saddle point problem. We search for a saddle point and then by the next theorem, this saddle point is a solution of the static problem. We are in an complete market, so by definition there exists exactly one measure $Q_0 \in \mathcal{M}_e^q$. We first formulate the theorem for the complete case to emphasize the idea of the approach. Afterwards, we formulate the theorem from convex analysis and adjust it to the stochastic context. The result is given in the next theorem:

Theorem 3.1.1. *Let $O = \{X \in L^p(P) : |E(U(X))| < \infty\} \subset L^p$. Suppose there exists a $y \in \mathbb{R},\ y \geq 0$, and an $X_0 \in O$ such that the Lagrangian $\tilde{L}(X, y) = L^*(X, yZ_0) = E(U(X)) - yE(Z_{Q_0}X - x)$ possesses as saddle point at X_0, y_0 i.e.,*

$$\tilde{L}(X_0, y) \geq \tilde{L}(X_0, y_0) \geq \tilde{L}(X, y_0), \tag{3.1}$$

for all $X \in O,\ y \geq 0$. Then X_0 solves:

$$\textit{maximize } E(U(X)) \quad \textit{s.t.} \quad E(Z_{Q_0}X) \leq x,\ X \in O.$$

The proof follows directly from the general theorem from convex analysis (Theorem 3.1.3).

3.1.2 Stochastic Analysis Using Methods from Convex Analysis

In this subsection, we apply results from convex analysis to the static problem. We introduce some notation and continue with the corresponding results from convex analysis. Afterwards, we put the results into the context of stochastic analysis. We define the Lagrangian of the constraint problem. Subsequently, we show that a saddle point of the Lagrangian directly leads to a solution of the constraint problem. We first have to take into account that using the usual $\leq$-relation in $\mathbb{R}$ leads to an infinite number of restriction. We have to define a $\leq -$ relation appropriately to this case:

Let Λ a linear space and its Λ^* dual space. A subset K of Λ is called a cone (with vertex at the origin) if it satisfies the condition: $x \in K \Rightarrow \forall \alpha \geq 0 : \alpha x \in K$ (see also Luenberger (1969), p.18). Let K be convex. For $x, y \in \Lambda,\ x \geq y$ means $x - y \in K$. K is then called the positive cone of Λ. Further, we introduce the corresponding cone in the dual space:

$$x^* \in K^{\oplus} = \{y^* \in \Lambda^* :\ \forall x \in K\ (x, y^*) \geq 0\} \Leftrightarrow: x^* \geq 0 \tag{3.2}$$

Note, dual pairs are denoted by $(\cdot,\cdot)$ $(x^*(x) =: (x, x^*)$ with $x \in L^p$ and $x^* \in L^q)$.

Lemma 3.1.2. *If K is closed in the normed vector space Λ, then $\tilde{K} = \{x \in \Lambda :\ \forall x^* \in K^{\oplus}\ (x, x^*) \geq 0\} = (K^{\oplus})^{\oplus} = K$.*

See Luenberger (1969), p.215 for a proof. K is convex by definition, so $K \supset (K^{\oplus})^{\oplus}$ always holds.

We continue with the formulation of the result from convex analysis and translate it to the stochastic setting. The result is taken from Luenberger (1969), p.221 Theorem 2:

Theorem 3.1.3. *Let h be a real-valued functional defined on a subset O of a linear vector space $\mathbb{L}$. Let G be a mapping from O into the normed space Λ having a nonempty positive cone K. Assume that K is closed. Suppose there exists a $\lambda_0^* \in \Lambda^*,\ \lambda_0^* \geq 0$, and an $X_0 \in O$ such that the Lagrangian $L(X, \lambda^*) = h(X) + (G(X), \lambda^*)$ defined on $O \times \Lambda^*$ possesses as saddle point at (X_0, λ_0^*); i.e.,*

$$L(X_0, \lambda^*) \leq L(X_0, \lambda_0^*) \leq L(X, \lambda_0^*), \tag{3.3}$$

for all $X \in O,\ \lambda^* \geq 0$. Then X_0 solves:

$$\textit{minimize } h(X) \quad \textit{s.t.} \quad G(X) \leq 0,\ X \in O.$$

Complete Case

We describe the reformulation to the saddle point problem in the complete case. We now establish Theorem 3.1.1. It is a direct consequence of Theorem 3.1.3, we simply set:

$$\begin{aligned} \Lambda \triangleq \Lambda^* \triangleq \mathbb{R}, &\quad K \triangleq \mathbb{R}^+ = K^{\oplus},\ \mathbb{L} = L^p(P), \\ h(X) \triangleq -E(U(X)), &\quad O \triangleq \{X \in L^p(P) : |E(U(X))| < \infty\} \\ G : L^p \to \Lambda = \mathbb{R}, &\quad G(X) \triangleq E_{Q_0}[X - x],\ x \in \mathcal{E} = \mathbb{R}^+ \end{aligned}$$

Q_0 denotes the equivalent martingale measure.

If (X_0, λ_0^*) (here: $\lambda^* = y$, this corresponds to $\lambda^* = y \cdot Z_{Q_0},\ y \in \mathbb{R}^+, Z_{Q_0} = \frac{dQ_0}{dP}$ in the framework of the incomplete case) is a saddlepoint of

$$\tilde{L}(X, \lambda^*) := -h(X) - (G(X), \lambda^*) = E(U(X)) - y(E(Z_{Q_0}X) - x), \quad (3.4)$$

then by Theorem 3.1.3 X_0 solves:

$$\max E(U(X)),\ s.t.\ E_{Q_0}(X) \leq x,\ X \in \mathcal{O} \quad (3.5)$$

Incomplete Case

We generalize the result in the complete case to the incomplete case. In a complete market, the Lagrange multiplier λ^* is just a real number or better it corresponds to a real number times the density of the unique martingale measure Q_0. In an incomplete market, we have more than one martingale measure (but we assume at least one), so it seems to be reasonable to define the Lagrange multipliers in the same form, a real number times a martingale measure. Further, we want to prove (3.3) only for Lagrange multipliers of this form. Consequently, we have to define a $\geq$-relation appropriately, i.e. we have to find the right positive cone. We start with $\mathbb{L}$. $\mathbb{L}$ has to be a normed linear vector space, so we can choose L^p. Further, we set $\Lambda \triangleq L^p,\ \Lambda^* \triangleq (L^p)^* = L^q$. So $\mathbb{L}$ and Λ coincide in this case.

Next note, from Riesz representation theorem, we know that for every $F \in (L^2)^*$ there exists an $\lambda^* \in L^2$ such that $F(X) = E(\lambda^* X)$. We identify an element $F \in L^2$ with the corresponding λ^*. However, we work in an L^p-setting. In this case, the mapping $F_\nu : L^p \to \mathbb{R}, \eta \mapsto \int \nu\eta dP,\ \nu \in L^q$ is also linear and continuous by Hölder's inequality. Further, it is well-known (see e.g. Alt (1992), Theorem 4.13 or Folland (1999), Proposition 6.13 and Theorem 6.15) that all functionals in $(L^p(P))^*$ are of the form $F_\nu(X) = E(\nu X)$, thus we have for $L^p,\ 1 \leq p < \infty$:

$$(L^p(P))^* = \{F_\nu | F_\nu : L^p \to \mathbb{R}, \eta \mapsto \int \nu\eta dP, \nu \in L^q\}$$

Hence, we can again identify ν with $F_\nu(\eta) = E(\nu\eta)$. In our case $\nu = \lambda^*$.

We define the included functions and spaces:

$$\Lambda \triangleq L^p(P), \qquad \Lambda^* = (L^p)^*(P) = L^q(P), \quad (3.6)$$

$$h(X) \triangleq -E(U(X)), \quad X \in O,\ O \triangleq \{X \in L^p : |E(U(X))| < \infty\} \quad (3.7)$$

$$G : Y = L^p \to \Lambda, \qquad G_x(X) = G(X) \triangleq X - x,\ x \in \mathcal{E} = \mathbb{R} \quad (3.8)$$

To establish that $E(ZX) \leq x$, we have to define the correct positive cone. We choose

$$K := \{X \in L^p(P) : \forall Z \in \mathcal{M}^q_{a,Z},\ y \geq 0,\ (X, F_{y,Z}) = E(yZX) \geq 0\}$$

With this definition $-G(X) = x - X \geq 0$ holds if and only if $\forall Z \in \mathcal{M}^q_{a,Z}$: $E(Z(x - X)) \geq 0$, which is equivalent to the desired constraint $E(ZX) \leq x$. Since the Lagrangian in the stochastic version only includes martingale measures up to standardization, we obtain the following bipolar relation:

$$K^{\oplus} = D, \text{ where } D := \{\lambda^* : \ \lambda^* = y \cdot Z, \ y \in \mathbb{R}^+, \ Z \in \mathcal{M}^q_{a,Z}\}. \tag{3.9}$$

To find a saddlepoint, it remains to find a pair (X_0, λ_0^*) such that equation (3.3) holds for all $\lambda^* \in D$. We then apply Theorem 3.1.3 and obtain an optimal solution. However, this approach works for a very general class of spaces. So it would be surprising, if we were always be able to find a solution using this way in any case. A pair (X_0, λ_0^*) even does not have to exist. The approach just describes a method to find a solution. It can happen that a saddlepoint $((X_0, \lambda_0^*))$ does not exist, we therefore cannot find an optimal solution using this method. We required that the densities have to be in L^q, since the terminal values are in L^p by assumption. But it is not ensured that e.g. $\mathcal{M}_a \neq \emptyset$. It can happen that $K^{\oplus}$ is empty. Fortunately, in section 4.2, we see that the solution exists in $\mathcal{M}^q_S$ for a special class of functions and under an additional assumption also in $\mathcal{M}^q_e$. We can apply the method described in this section to find an explicit solution.

To show (3.9) note that D is closed. Thus by Lemma 3.1.2 $((D)^{\oplus})^{\oplus} = D$. Further, we have defined K such that $K = D^{\oplus}$ and hence $K^{\oplus} = ((D^{\oplus})^{\oplus}) = D$. Finally, notice that we maximize $-h(X) = E(U(X))$ and we therefore define as our Lagrangian:

$$\begin{aligned} \tilde{L}(X, \lambda^*) &= -h(X) - (G(X), F) && (3.10)\\ &= E(U(X)) - E(\lambda^*(X - x)) \\ &= E(U(X)) - y(E(ZX) - x) && (3.11) \end{aligned}$$

where $\lambda^* = y \cdot Z, \ y \in \mathbb{R}, Z \in \mathcal{M}^q_{a,Z}$.

With (3.11), (3.3) converts to:

$$\tilde{L}(X_0, \lambda^*) \geq \tilde{L}(X_0, \lambda_0^*) \geq \tilde{L}(X, \lambda_0^*), \tag{3.12}$$

for all $X \in O, \ \lambda^* \geq 0$ or $\lambda^* \in D$, respectively. We summarize in the following corollary:

If (X_0, λ_0^*) (remember: $\lambda^* = y \cdot Z, y \in \mathbb{R}^+, Z \in \mathcal{M}^q_{a,Z}$) is a saddlepoint of

$$\tilde{L}(X, \lambda^*) = -h(X) - (G(X), \lambda^*) = E(U(X)) - y(E(ZX) - x), \tag{3.13}$$

we have by Theorem 3.1.3 that X_0 solves:

$$\max E(U(X)), \ s.t. \forall Q : \ E_Q(X) \leq x, \ X \in \mathcal{O} \tag{3.14}$$

Corollary 3.1.4. *Suppose there exists a* $y_0 \geq 0$ *and a* $Z_{opt} \in \mathcal{M}^q_{a,Z}$ *and an* $X_0 \in O$ *such that the Lagrangian* $\tilde{L}(X, y \cdot Z) = E(U(X)) - y(E(ZX) - x)$ *possesses a saddle point at* $(X_0, y_0 \cdot Z_{opt})$*, i.e.*

$$\tilde{L}(X_0, y \cdot Z) \geq \tilde{L}(X_0, y_0 \cdot Z_{opt}) \geq \tilde{L}(X, y_0 \cdot Z_{opt}),$$

for all $X \in O,\ y \geq 0, Z \in \mathcal{M}^q_{a,Z}$ *or* $yZ = \lambda^* \in D$*, respectively. Then* X_0 *solves:*

$$\max E(U(X)),\ s.t. \forall Q \in \mathcal{M}^q_a:\ E_Q(X) \leq x,\ X \in \mathcal{O} \tag{3.15}$$

Note, if we already have that $EU(X) < \infty$ for $X \in L^p$, a static solution is also a dynamic one by Theorem 2.2.10.

3.2 Saddlepoint Problem

Our next goal is to find a saddle point of the Lagrangian. General existence results can be found in Kramkov and Schachermayer (1999) for utility functions of type one and Schachermayer (2001) for utility functions of type two under the assumption of reasonable asymptotic elasticity, as defined in section 3.2.1 and 3.2.2. All authors work with rather weak assumptions, so the Lagrange concept and the transformation from the dynamic to the static problem (from $\mathcal{X}, \mathcal{Y}$ to C, D in their notation) is hidden within refined technical proofs leading to slightly different results. Fortunately, there are connections to our approach. The goal of this section is to explain the used concepts and to provide the reader with an explicit solution method rather than a technical existence result. In principal Kramkov and Schachermayer (1999) rely on an imitation of the minimax-theorem, i.e. under suitable assumption there exists a pair $(\tilde{u}, \tilde{p})$ such that $L(\tilde{u}, \tilde{p}) = \min_A \max_B L(u, p) = \max_B \min_A L(u, p)$, see e.g. Luenberger (1969) p.208. The pair $(\tilde{u}, \tilde{p})$ is then a saddle point, since by Ekeland and Temam (1976) chapter VI, it holds:

Proposition 3.2.1. *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be arbitrary sets and* L *a real-valued function defined on* $\mathbb{A} \times \mathbb{B}$*.* L *possesses a saddle point* $(\tilde{u}, \tilde{p})$ *if and only if*

$$L(\tilde{u}, \tilde{p}) = \min_{u \in \mathbb{A}} \sup_{p \in \mathbb{B}} L(u, p) = \max_{p \in \mathbb{B}} \inf_{u \in \mathbb{A}} L(u, p)$$

So proving a minimax-theorem or looking for a saddle point directly are equivalent. We try to find a saddlepoint directly:

Method to Find a Saddlepoint

Next, we describe a method how to find a saddlepoint of $\tilde{L}$. We proceed by fixing λ_1^* and look for a $X_0(\lambda_1^*)$, which satisfies the second inequality of (3.12), i.e.

$$X_0(\lambda_1^*) = \arg\max_X \tilde{L}(X, \lambda_1^*). \tag{3.16}$$

To find the solution of (3.16) ($X_0(\lambda^*)$), we introduce the convex dual of the chosen utility function U defined on a still arbitrary domain $\mathbb{D}$ (Fenchel-Legendre). It arises canonically from (3.11):

$$\check{U}(y) := \sup_{x\in\mathbb{D}}[U(x) - xy] \tag{3.17}$$

It is defined for all y with $\sup_{x\in\mathbb{D}}[U(x) - xy] < \infty$. If U is continuously differentiable and strictly convex, then U' is invertible and the supremum of (3.17) is attained at $I(y) = (U')^{-1}$ and we have:

$$\check{U}(y) = U(I(y)) - I(y)y \tag{3.18}$$

We fix a $\lambda_1^* = Z_1 \cdot y_1$, to solve (3.16):

$$\begin{aligned}\tilde{L}(X, \lambda_1^*) &= E(U(X)) - (E(\lambda_1^* X - \lambda_1^* x)) = E(U(X) - \lambda_1^* X) + xy_1 \\ &\leq E(\check{U}(\lambda_1^*)) + xy_1\end{aligned}$$

Equality holds if and only if:

$$X_0(\lambda_1^*) = I(\lambda_1^*) = I(Z_1 \cdot y_1) \tag{3.19}$$

Next, we choose a λ_1 such that $(\lambda_1^*, X_0(\lambda_1^*))$ also satisfies the first inequality, i.e.

$$\begin{aligned}\forall\lambda^* \geq 0: \ \tilde{L}(X_0(\lambda_1^*), \lambda^*) &\geq \tilde{L}(X_0(\lambda_1^*), \lambda_1^*), \\ \Leftrightarrow \tilde{L}(X_0(\lambda_1^*), \lambda_1^*) &= \min_{\lambda^*\geq 0} \tilde{L}(X_0(\lambda_1^*), \lambda^*) \\ \Leftrightarrow E(U(X_0(\lambda_1^*))) - (E(\lambda_1^*(X_0(\lambda_1^*) - x)) &= \min_{\lambda^*\geq 0} E(U(X_0(\lambda_1^*))) \\ &\quad -(E(\lambda^*(X_0(\lambda_1^*) - x)))\end{aligned} \tag{3.20}$$

So according to (3.16) to obtain an optimal X_0, we further have to solve:

$$\min_{y_1\geq 0, Z_1} \phi(y_1, Z_1) \tag{3.21}$$

where

$$\phi(y_1, Z_1) = E(U(I(Z_1 \cdot y_1)) - Z_1 \cdot y_1 I(Z_1 \cdot y_1)) + xy_1$$

$\lambda_0^* = y_0 Z_{opt}$ denotes the optimal solution of (3.21). So $(\lambda_0^*, X_0(\lambda_0^*))$ is a saddlepoint, provided that $X_0(\lambda_0^*) \in L^p(P)$:

$$X_0(\lambda_0^*) = I(y_0 Z_{opt}) \in L^p(P) \tag{3.22}$$

is an optimal solution of the primal problem.

Alternatively, the following approach is possible, we can extend (3.20):

$$\begin{aligned} E(U(X_0(\lambda_1^*))) - (E(\lambda_1^*(X_0(\lambda_1^*) - x)) &= \min_{\lambda^* \geq 0} E(U(X_0(\lambda_1^*))) \\ &\quad -(E(\lambda^*(X_0(\lambda_1^*) - x))) \qquad (3.23) \\ \Leftrightarrow (E(\lambda_1^*(X_0(\lambda_1^*) - x)) &= \max_{\lambda^* \geq 0}(E(\lambda^*(X_0(\lambda_1^*) - x)) = (*) \end{aligned}$$

The last equivalence only holds, if $E|U(X_0(\lambda_1))| < \infty$. Since the constraints have to be satisfied, $(*)$ is smaller or equal to zero. So we look for a candidate λ_0^* which either satisfies $\lambda_0^* = 0$ or uniquely solves

$$\mathcal{X}(\lambda_0^*) \equiv E(Z_{\lambda_0^*} X_0(\lambda_0^*)) = x, \qquad (3.24)$$

where $\lambda_0^* = y_{\lambda_0^*} \cdot Z_{\lambda_0^*}$. This approach is appropriate in an complete market. $\lambda_0^* = \mathcal{Y}(x) Z_{Q_0}$, where $\mathcal{Y}$ is the inverse function of $\mathcal{X}$. It is also used in the incomplete case, but after carrying out another minimization problem of the Z-part, such that the optimal measure is already known only dependent on y_0 $(Z(y_0))$. We then look for a solution of (3.24) of the form $\lambda_0^* = \mathcal{Y}(x) Z(\mathcal{Y}(x))$. If $Z(y)$ is independent of y, we can proceed as in the complete case. The approach in the incomplete case is explained in detail in section 3.2.2. So consider the complete case. We again have to check if $X_0(\lambda_0^*) \in L^p(P)$. This implies that $\mathcal{X}$ as defined in (3.24) is well-defined since $Z_{Q_0} X_0 \in L^1$ $(Z_{Q_0} \in L^q)$. If for $\lambda_0^* = 0$, $X_0(\lambda_0^*) \notin L^p(P)$ -what often appears, we turn towards the second condition (3.24). In the second case, under appropriate conditions on the chosen utility function, $\mathcal{X}$ is invertible and the inverse function $\mathcal{Y} = \mathcal{X}^{-1}$ is regular enough, such that $X_0(\mathcal{Y}(x)) \in O$. If $\mathcal{Y}(x) \geq 0$, then we take λ_0^* and see that both inequalities are satisfied, i.e. we have a saddlepoint. Hence $X_0(\mathcal{Y}(x))$ is an optimal solution.

Summarizing, we need the following properties to establish an optimal solution:

Necessary Properties of X_0 and $\mathcal{X}$:

First note, if we have a strictly increasing utility function, loosen the budget constraints (increasing the initial wealth) always leads to an increase of the optimal value. The Lagrange multiplier is strictly bigger than zero and the constraint is satisfied with equality. In these cases $(0, I(0))$ does not possess the right integrability or at least the value of both possible solutions is equal. We now want to explicitly calculate $\mathcal{X}$ and its inverse to obtain the value function V. We have to check if all necessary properties (3.2.1) of $\mathcal{X}$ are satisfied:

Property 3.2.1. *1. $\mathcal{X}$ is invertible*

2. $\mathcal{X}(y) < \infty, \; 0 < y < \infty$

3. *All $x \in \mathcal{E}$ have to be in the domain of $\mathcal{Y}$*

4. $X_0(x) := I(\mathcal{Y}(x)Z_{\mathcal{Y}(x)})$ *has to be an allowable wealth process, i.e.* $X_0(x) \in L^p$

5. $\mathcal{Y}(x) \geq 0$

The first three items are necessary to ensure that $\mathcal{Y}$ exists on the right domain and maps into $\mathbb{R}$. Four guarantees that the optimal solution is within the allowed region. Finally, the last item ensures that the Lagrange multiplier is non-negative as required in Theorem 3.1.3. Again U is assumed to be concave, so $O = L^p$.

Duality

Next, we give a slight idea how the duality relation could like. Analogously to Luenberger (1969), we call ϕ the dual functional of the value function V_0 as defined in (2.24) with $\xi \equiv 0$ ($V_{\xi=0} = V_0$ and usually we leave out the index 0). More precisely,

$$\begin{aligned} V_0(x) &= \sup_{X \in O, G_0(X) \leq x} E[U(X)] \\ &\triangleq -\omega(x) \end{aligned}$$

where $\omega(z) = \inf\{f(X) : G_0(X) \leq z, X \in O\}$ the primal functional from convex analysis and

$$\phi(y_1, Z_1) \triangleq -\varphi(\lambda^*)$$

where $\varphi(\lambda^*) := \inf\{h(X) + (G(X), \lambda^*),\ X \in O\}$ is the dual functional. We now cite two results from convex analysis again taken from Luenberger (1969). The first result can be applied without further adjustments. The second one states the Lagrange-duality, which cannot be translated so easily at least in the incomplete case, but it gives a good intuition how it shall look like in the stochastic case.

Proposition 3.2.2. *The dual functional is convex and it is equal to:*

$$\phi(y_1, Z_1) = \sup_{x \in \Gamma}[V_0(x) - xy_1],$$

where $\Gamma = \{x : \exists X \in O : G(X) \leq x\}$

Proof. The dual in Luenberger (1969) is convex and $\varphi(\lambda^*) = \inf[\omega(x) + (x, \lambda^*)]$. Note, $xy_1 = xy_1 E(Z_1)$. □

Next, we state the Lagrange duality:

Theorem 3.2.3. *Let h be a real-valued convex functional defined on a convex set O of a vector space L, and let G be a convex mapping of L into a normed vector space Λ. Suppose there exists an X_1 such that $G(X_1) < 0$ and that $\omega(0) < \infty$. Then*

$$\inf_{G(X)\leq 0, X\in O} h(X) = \max_{\lambda^*\geq 0} \varphi(\lambda^*) \tag{3.25}$$

the maximum on the right is achieved by some $\lambda_0^* \geq 0$. If the infimum on the left is attained by some $X_0 \in O$, then $(G(X_0), \lambda_0^*) = 0$ and X_0 minimizes $h(X) + (G(X), \lambda_0^*), x \in O$.

In the complete case, the existence of such a X_1 is obvious. However, in the incomplete case that it is not. We have to ensure that there exists a X_1 such that $G(X_1) < 0$ with respect to the cone K. That means $G(X_1) = X_1 - x$ has to be an interior point of $-K$. Since $X_1 \in L^p$ is arbitrary, we have to prove that K contains inner points. If this assertion is true certainly depends on the structure of the set of martingale measures. We set without loss of generality $x = 0$. We go on constructing an example, where the interior is empty:

$$\forall f \in -K \; \forall \epsilon > 0 \; \exists A \in B(f, \epsilon) \wedge A \in (-K)^c : \tag{3.26}$$

Example 3.2.1. Recall, $-K$ is given by:

$$-K = \{f \in L^p(P) : \forall Q \in \mathcal{M}_a^q,\ y \geq 0 \ \ E_Q(yf) \leq 0\}$$

Suppose, our stock is already a martingale $S = S_0 + M$ on the probability space $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}, \lambda)$, and filtration $(\mathcal{F}_t)_t$ where λ is the Lebeque-measure and $\mathcal{B}$ the Borel-σ-Algebra and a filtration $(\mathcal{F}_t)_t$ with $[0, \frac{1}{n}] \in \mathcal{F}_0$ for all $n \in \mathbb{N}$, e.g. $S_t = g$ for all t and g a uniformly distributed random variable and $\mathcal{F}_t = \mathcal{B}$. So $P([0, \frac{1}{n}]) > 0$ for every fixed n, but $P([0, \frac{1}{n}[0,1]) \to 0$ for n tending to infinity and $P(\{0\}) = 0$. We further define probability measures Q_n:

$$Q_n(A) = \int_A 1_{[0,\frac{1}{n}]} P([0, \frac{1}{n}])^{-1} dP,\ n \in \mathbb{N}$$

Note, $Q_n([0, \frac{1}{n}]) = 1$ and $Q_n \ll P$. Further, we have that $Q_n \in \mathcal{M}_a^q$, since

$$E_{Q_n}(S_t|\mathcal{F}_u) = E(S_t|\mathcal{F}_u) = S_u \quad Q_n - a.s.$$

Next, we fix an arbitrary $f \in -K$. We have to find for every $\epsilon = \frac{C}{n}$, $C \in (0, \infty)$, $n \in \mathbb{N}$, an A_n such that $\|A_n - f\|_{L^p(P)} \leq \frac{C}{n}$ and $A_n \in (-K)^c$. We set:

$$A_n(\omega) = 1_{\{\omega \in [0,\frac{1}{n}]\}} + 1_{\{\omega \in (\frac{1}{n},1]\}} f$$

Hence,

$A_n \to f \;\; P - a.s.$, since $P(\{0\}) = 0$. Further, since (A_n) are trivially equi-integrable ($h = 1 + |f|$ using Theorem 20.4 in Bauer (1968)), we have that $A_n \stackrel{L^p(P)}{\to} f$. Further, we have:

$$E_{Q_n}(A_n) = Q_n([0, \frac{1}{n}]) = 1 > 0$$

Thus, there exists a $Q_n \in \mathcal{M}_a^q$ and a $y = 1 \geq 0$ such that $E_Q(yA_n) > 0$, i.e. $A_n \in (-K)^c$. (3.26) is true, $-K$ has an empty interior.

■

So example 3.2.1 shows that a direct application of Theorem 3.2.3 is not possible. Even more obvious, if the market is totally incomplete, i.e. we can only trade the bond, then every absolutely continuous measure is a martingale measure. The general theorem from convex analysis cannot be used. Clearly, the set of martingale measures will play an important role. However, under appropriate assumptions we can expect the following result:

Let $V_0(0) < \infty$. Then

$$\sup_{X \in O:\, \forall Z \in \mathcal{M}_{a,Z}^q \; E(ZX) \leq x} E(U(X)) = \min_{y \geq 0, Z \in \mathcal{M}_{a,Z}^q} \phi(y, Z) \tag{3.27}$$

the maximum on the right is achieved by some $y_0 \geq 0$ and $Z_{opt} \in \mathcal{M}_{a,Z}^q$. If the infimum on the left is attained by some $X_0 \in O$, then $E(y_0 Z_{opt} X_0) = x$ and X_0 maximizes $E(U(X)) - yE(ZX - x), x \in \mathcal{E}$.

Whereas the result is not true in a totally incomplete market, it is true in a complete market.

We have to take particular care upon the assumption $\omega(0) < \infty$. If this is not the case, the maximum on the right hand side might not be admitted.

At first, we treat (3.21) for the complete case in the next subsection and afterwards we generalize the result to the incomplete case. Duality results easily follow in most cases.

3.2.1 Complete Case and Examples

As mentioned, in the complete case a lot of things simplify extremely. So we devote this section to the solution of the saddle point problem and the dual relation in a complete market:

Method in the Complete Case

To find a saddle point, firstly we fixed λ_1^* or y_1 in this case. Then we solve for $X_0(\lambda_1^*)$:

$$X_0(\lambda_1^*) = \arg\min_X L(X, \lambda_1^*) = I(Z_{Q_0} \cdot y_1). \tag{3.28}$$

The minimization over the measures in (3.20) can be omitted. We directly start with (3.24). If $\mathcal{X}$ is good enough, set

$$\mathcal{Y} = \mathcal{X}^{-1},$$

where

$$\mathcal{X}(\lambda_1^*) = E(Z_{\lambda_1^*} X_0(\lambda_1^*)) = E(Z_{Q_0} I(Z_{Q_0} y_1)).$$

As before, a saddle point $(y_1, X_0(y_1))$ of

$$E(U(X_0(y_1)) - Z_{Q_0} \cdot y_1 X_0(y_1)) + x y_1 \tag{3.29}$$

is given (provided $X_0(y_1) \in \mathcal{W})$), by either $(0, I(0))$ or $(\mathcal{Y}(x), I(Z_{Q_0}\mathcal{Y}(x)))$. The resulting value function $V_0(x)$ of the second case is of the form:

$$V_0(x) = G_0(\mathcal{Y}(x)),\ G_0(y) = E(U(I(yZ_0))) \tag{3.30}$$

since

$$(3.29) = E(U(I(\mathcal{Y}(x)Z_{Q_0}))) - \mathcal{Y}(x)E(I(Z_{Q_0}\mathcal{Y}(x))) + \mathcal{Y}(x)x \tag{3.31}$$

By definition $E(Z_{Q_0} I(Z_{Q_0}\mathcal{Y}(x))) = \mathcal{X}(\mathcal{Y}(x)) = x$ and assertion (3.30) follows.

Duality in the complete Case

Let's assume $\mathcal{X}^{-1}$ exists. Then, the above result leads us very easily to a duality result of the form (3.27), where the measure is fixed:

$$\begin{aligned}
\phi(y_1, Z_{Q_0}) &= E(\check{U}(y_1, Z_{Q_0})) + x y_1 \\
&= E(\max_X (U(X) - Z_{Q_0} \cdot y_1 X)) + x y_1 \\
&= E(U(X_0(y_1)) - Z_{Q_0} \cdot y_1 X_0(y_1)) + x y_1 \\
&\geq E(U(X_0(\mathcal{Y}(x))) - Z_{Q_0} \cdot y_1 X_0(\mathcal{Y}(x))) + x y_1 \\
&= V(x)
\end{aligned} \tag{3.32}$$

where

$$\mathcal{Y}(x) = \begin{cases} \mathcal{X}^{-1}(x) & \mathcal{X}^{-1}(x) > 0 \\ 0 & \mathcal{X}^{-1}(x) \leq 0 \end{cases} \tag{3.33}$$

The last equality of (3.32) is analogous to (3.31). The inequality in (3.32) is an equality if and only if $y_1 = \mathcal{Y}(x)$. Hence, we have derived the desired duality result:

$$\min_y \phi(y, Z_{Q_0}) = V(x) = \min_{X:\, E(Z_{Q_0}X) \leq x} E(U(X)). \tag{3.34}$$

We now apply the suggested method to some examples, by proving property 3.2.1:

Utility Functions of Type I

In this subsection, we discuss a class of utility functions which is used to derive a solution in Karatzas (1996) and Kramkov and Schachermayer (1999). In section 1.3.2, we called it utility functions of type I:

Definition 3.2.1. We call a utility function $U : (0, \infty) \to \mathbb{R}$ of type I, if U is strictly increasing, strictly concave and twice continuously differentiable with $U'(\infty) := \lim_{x\to\infty} U'(x) = 0$ and $U'(0+) := \lim_{x\downarrow 0} U'(x) = \infty$.

The last condition is usually called Inada-condition. Sometimes it is also assumed that $U(x) = -\infty$ for $(-\infty, 0]$. It suffices to consider only continuous differentiable functions. The class includes power utility function with $p \in (0, 1)$ and the logarithmic utility function. Recall the convex dual of U is defined as $\check{U}(y) = \sup_{x\in D}(U(x) - xy),\ D = (0, \infty)$. Since U is strictly concave, its derivative is invertible. Let's called it $I = (U')^{-1}$. It possesses the following properties:

Lemma 3.2.4. *For a utility function of type I we have: U' and its inverse I map on $(0, \infty)$ and both are strictly decreasing. The derivative of I equals $\frac{1}{U''(I(y))}$. Further, the following holds: $\check{U}$ is convex decreasing and continuously differentiable function and*

$$\check{U}(y) = UI(y) - yI(y), \qquad \check{U}'(y) = -I(y) \qquad 0 < y < \infty \tag{3.35}$$

$$U(x) = \min_{0<y<\infty}(\check{U}(y) + xy) \quad = \check{U}(U'(x)) + U'(x)x, \quad 0 < x < \infty \tag{3.36}$$

$$\check{U}(\infty) = U(0+), \qquad \check{U}(0+) = U(\infty) \tag{3.37}$$

Proof. U' is strictly decreasing because of the concavity $(U''(\cdot) < 0)$. Since $U'' < 0$, the inverse function theorem can be applied. The form of the derivative and the strictly decrease of I follows immediately. Since both functions are strictly decreasing and continuous and $I(\infty) = U'(\infty) = 0$ and $I(0+) = U'(0+) = \infty$, both map on $(0, \infty)$. Since U is differentiable and concave $U(\cdot)$ attains its maximum at $I(y)$. The first equality of (3.35) follows. The differentiability of I yields the differentiability of $\check{U}$. Simple differentiation gives the second equality, and the convexity. (3.36) follows analogously. The last equality follows from the two previous ones and the properties of U' and I. □

For simplicity, we use the Brownian model. Note, $(0, I(0+))$ is no possible solution, since $I(0+) = \infty$. Applying Lemma 3.2.4, we can now try to prove property 3.2.1. To establish property 2. and 4., we impose one more condition, see also Karatzas (1996), p.40:

Assumption 3.2.1.

$$\exists s \in (0, \infty)\, \exists K : I(y) \le K y^{-s}, \tag{3.38}$$

and $\theta()$ is uniformly bounded.

In a complete market and if σ and μ are bounded, then $\theta = \sigma^{-1}\mu$ is automatically bounded. Assumption (3.2.1) includes power utilities $-\infty < p < 0$, since $I(y) = y^{\frac{-1}{1-p}}$ and hence $s \in (0,1)$. Power utilities with $p \in (0,1)$, correspond to $s > 1$. Using assumption 3.2.1, we have for $s \in (0,1)$:

$$\begin{aligned}\mathcal{X}(y) &= E(Z_{Q_0}X_0(y)) = E(Z_{Q_0}I(Z_{Q_0}y)) \\ &= E(Z_{Q_0}^{1-s})y^{-s} \leq y^{-s}(E(Z_{Q_0}))^{1-s} = y^{-s}\end{aligned}$$

The last inequality uses the concavity of $x^{1-s}, s \in (0,1)$ and Jensen's inequality. To establish property 2 also for $p > 1$, we use that θ is bounded, then $E(Z_{Q_0}^{1-s}) \in L^1(P)$, since for $p \in \mathbb{R}$, we have

$$\begin{aligned}|Z_p| = |Z^p| &= \exp\{-\int_0^T p\theta_s dW_s - \frac{1}{2p}\int_0^T p^2\|\theta_s\|^2 ds\} \\ &= \exp\{-\int_0^T p\theta_s dW_s - \frac{1}{2}\int_0^T p^2\|\theta_s\|^2 ds + \frac{p-1}{2p}\int_0^T p^2\|\theta_s\|^2 ds\} \\ &\leq C(p)\exp\{-\int_0^T p\theta_s dW_s - \frac{1}{2}\int_0^T p^2\|\theta_s\|^2 ds\} \qquad (3.39)\end{aligned}$$

$C(p) \in (0,\infty)$ a constant, depending on θ. This is satisfied since θ assumed to be bounded. Further, Novikov's condition is satisfied, thus

$$\tilde{Z}_t^p := \exp\{-\int_0^t p\theta_s dW_s - \frac{1}{2}\int_0^t p^2\|\theta_s\|^2 ds\}$$

is a martingale and $E(\tilde{Z}_T^p) = 1$. Property 3.2.1.2. follows. Further, $\mathcal{X}(y) = E(Z_{Q_0}I(yZ_0))$ is continuous and strictly decreasing and because of $\mathcal{X}(0+) = \infty$, $\mathcal{X}(\infty) = 0$, and so $\mathcal{X}$ maps onto $(0,\infty)$. Thus, 1., 3., and 5. are satisfied. Finally, we have to establish the strategies are allowed, i.e. that $X \in L^p(P)$. (Karatzas (1996) gives a proof for the usual admissible portfolios - portfolios with a finite credit line). We give a prove for $L^p(\lambda \otimes P)$ strategies. We again use assumption 3.2.1:

$$E(X_0^p(y)) = E(I^p(Z_{Q_0}y)) \leq E(Z_{Q_0}^{-ps})K_1 y^{-ps}$$

The p-integrability follows as in equation (3.39). We know from the reformulation that a p-integrable portfolio exists. The same argument follows in a semimartingale model with $[N]_T$ bounded, where $Z_{Q_0} = \mathcal{E}_T(N)$. So the optimal solution is:

$$X_0(x) = I(\mathcal{Y}(x)Z_{Q_0}).$$

Exponential Utility Function

In this section, we prove the five properties of $\mathcal{X}$ for the exponential utility function:

Definition 3.2.2. The exponential utility function is of the form:

$$U_{\exp\ \alpha}(X) = 1 - e^{-\alpha X}$$

α is the degree of risk aversion.

The constant 1 does not play a role when searching for a minimizer. So we leave out the constant 1 in the sequel. However note, its job is to ensure that the utility is equal to zero, if the wealth process is equal to zero. That makes sense, if our main concern is to hedge a terminal payoff, i.e. $Y_T - \xi = X = 0$. Zero means the claim is perfectly hedged. A negative wealth gives negative utility, positive wealth yields a positive value of the utility function. So for a positive terminal wealth this is comparable to power utilities. If we define the utility of power functions minus infinity for negative wealth, we recognize a big advantage of exponential utility functions: A negative value of the wealth process (maybe very unlikely) does not immediately lead to a value of the utility function of minus infinity. We leave out the constant 1 to make calculations easier.

The class of exponential utility function is in $\mathcal{C}^\infty$, strictly increasing, strictly concave and:

$$\begin{array}{lll} U'(\infty) = 0, & U'(0) = \alpha, & U'(-\infty) = \infty \\ \check{U}(y) = U(I(y)) - yI(y), & \check{U}'(y) = -I(y) = -\frac{(-\log\frac{y}{\alpha})}{\alpha}, & y \in (0,\infty) \\ U(x) = \min\limits_{0<y<\infty}(\check{U}(y) + xy) & = \check{U}(U'(x)) + U'(x)x, & x \in \mathbb{R} \end{array}$$

Again $I(0+) = -\infty$ does not lead to an admissible solution. Also in this case, we have to impose an additional assumption:

Assumption 3.2.2. *The relative entropy of the equivalent martingale measure Q_0 with respect to the true measure is finite, i.e. $H(Q_0|P) < \infty$, where*

$$H(Q_0|P) = \left\{ \begin{array}{ll} E_P(\frac{dQ_0}{dP}\log\frac{dQ_0}{dP}), & \text{if } Q_0 \ll P \\ \infty, & \text{otherwise} \end{array} \right. .$$

Clearly, this assumption is satisfied in a complete Brownian Model with bounded coefficients, since θ is bounded, see also Lemma 3.2.9.

We start by calculating $\mathcal{X}$ for $0 < y < \infty$:

$$\begin{aligned} \mathcal{X}(y) &= E\left(Z_{Q_0}\frac{(-\log\frac{yZ_0}{\alpha})}{\alpha}\right) = -\frac{1}{\alpha}\left(\log(\frac{y}{\alpha} + E(Z_{Q_0}\log Z_{Q_0})\right) \\ &= -\frac{\log(\frac{y}{\alpha})}{\alpha} - \frac{1}{\alpha}H(Q_0|P) \end{aligned}$$

where $Z_{Q_0} = \frac{dQ_0}{dP}$. Hence, using assumption 3.2.2, $\mathcal{X}$ is finite. $\mathcal{X}(\infty) = -\infty$, $\mathcal{X}(-\infty) = \infty$, and since

$$\mathcal{X}'(y) = -\frac{1}{y\alpha} < 0, 0 < y < \infty,$$

$\mathcal{X}$ is strictly decreasing and the range of $\mathcal{X}$ is the real line. Property 3.2.1.1.-3. are satisfied. The next few lines are devoted to the fourth property:

We consider $\log Z_{Q_0}$. Z_{Q_0} is the density of the unique martingale measure. According to Theorem 1.2.5 it can be represented as a Girsanov functional with parameter $\theta = \sigma^{-1}\mu$. Since the market is complete σ is invertible. So we have using Young's inequality for the L^2-case :

$$\begin{aligned} E\log^2 Z_{Q_0} &= E(-\int_0^T \theta_t' dW_t - \frac{1}{2}\int_0^T \|\theta_t\|^2 dt)^2 \\ &\leq 2E(\int_0^T \theta_t' dW_t)^2 + 2E(\frac{1}{2}\int_0^T \|\theta_t\|^2 dt)^2 \\ &\leq E(\int_0^T \|\theta_t\|^2 dt) + 2E(\frac{1}{2}\int_0^T \|\theta_t\|^2 dt)^2 < \infty \end{aligned}$$

by isometry and the model assumptions of the parameters and C a positive constant. For L^p-portfolios, we obtain:

$$\begin{aligned} E\log^p Z_{Q_0} &= E(-\int_0^T \theta_t' dW_t - \frac{1}{2}\int_0^T \|\theta_t\|^2 dt)^p \\ &\leq E(-\int_0^T \theta_t' dW_t - C)^p \\ &\leq C_1 \cdot E((W_T - C_2)^p) < \infty \end{aligned}$$

since W_T is normally distributed and all moments exist! C_1, C_2 positive constant since θ is bounded in the Brownian Model with bounded coefficients. Even more generally.: $Z_{Q_0} = \mathcal{E}_T(N)$. Let $EN_T^{2p} < \infty$, then:

$$\begin{aligned} E\log^p Z_{Q_0} &= E(N_T - \frac{1}{2}[N]_T)^p \\ &\leq 2^{p-1}E(N_T^p + (\frac{1}{2})^p[N]_T^p) \\ &\leq C(p) \cdot E(N_T^{2p}) < \infty \end{aligned}$$

by the inequalities Burkholder-Davis-Gundy and Doob and $C(p)$ a positive constant dependent on p. Z_{Q_0} is the minimal entropy measure and so $EN_T^{2p} < \infty$ under the assumptions of Theorem 4.1.10. Since $E(\exp(-(-\log Z_0)) = 1$, the optimal solution is

$$X_0^{exp}(x) = -\log Z_0 + H(Q_0|P) + x \in \mathcal{O}$$

Finally, we want to determine the value function for this problem. We start inverting $\mathcal{X}$. By simple calculation, we obtain:

$$\mathcal{Y}(x) = \alpha \exp\{-\alpha x - H(Q_0|P)\} \tag{3.40}$$

Property 3.2.1.5. is satisfied, since $\alpha > 0$.

In the next step, we derive $G_0(y)$ from (3.30):

$$\begin{aligned} G_0(y) &= E(U(I(yZ_0))) = E\left(-\exp\left\{-\alpha\left(\frac{-\log\frac{yZ_0}{\alpha}}{\alpha}\right)\right\}\right) \\ &= -\frac{y}{\alpha}E(Z_{Q_0}) = -\frac{y}{\alpha} \end{aligned} \tag{3.41}$$

Thus, we obtain the corresponding value function:

$$V_{exp\,\alpha}(x) = G(\mathcal{Y}(x)) = -\frac{\alpha\exp\{-\alpha x - H(Q_0|P)\}}{\alpha} \tag{3.42}$$

$$= -\exp\{-\alpha x - H(Q_0|P)\} \tag{3.43}$$

Remark 3.2.1. Assumption 3.2.2 is crucial also in the incomplete case. In this case we choose the optimal measure in the class of equivalent (local) martingale measures $\mathbb{P}_f(P)$, which have finite relative entropy with respect to the true measure P (to P_ξ). In fact, provided $\xi \equiv 0$ as in this case, the optimal measure is the minimal entropy measure and we replace $H(Q_0|P)$ by $\inf_{Q\in\mathbb{P}_f(P)} H(Q|P)$. We will do almost the same calculations in the incomplete case, where we use the nice feature that for exponential utility functions the minimal entropy measure does not depend on the initial wealth.

Isoelastic-Utility Functions $\alpha > 1$

We mention iso-elastic functions to tie a connection between iso-elastic utility functions, the exponential utility function and there even partial sums. Further, we want to emphasize the role of superhedging strategies. We only consider the even ones, since they are at least concave, even neither decreasing nor increasing over the real line. This problem is fully solved in Leitner (2001), considering the absolute value of this functions. An explicit portfolio is obtained in Bürkel (2004) via a Ricatti-BSDE- approach. It is a generalization of the mean variance problem. Our only goal is to hedge the claim. This is rather boring, if the claim is zero. If we have a positive initial wealth, we consume everything and we have maximum utility. The constraint is not satisfied with equality, a hedging strategy would be suboptimal to this superhedging strategy and $\mathcal{Y} < 0$. The Lagrange multiplier is zero. If the initial wealth is negative, $\mathcal{Y}$ will be positive. The Lagrange multiplier will be strictly bigger than zero. Since $Y_T = X \equiv 0, E_Q(X) = 0$ is not hedgeable. We illustrate these claims by simply calculating $I, \mathcal{X}, \mathcal{Y}, G$:

$$I(y) = -y^{\frac{1}{\alpha-1}},\ I(\infty) = -\infty,\ I(-\infty) = \infty,\ y \in \mathbb{R} \tag{3.44}$$

$$\mathcal{X}(y) = -E(Z_{Q_0} y^{\frac{1}{\alpha-1}} Z_{Q_0}^{\frac{1}{\alpha-1}}) = -y^{\frac{1}{\alpha-1}} E(Z_{Q_0}^{\frac{\alpha}{\alpha-1}}) \equiv -y^{\frac{1}{\alpha-1}} D \tag{3.45}$$

(3.45) is strictly decreasing and finite for finite y. So property 3.2.1.1. and 3.2.1.2. hold. From (3.45), the inverse is easily determined:

$$\mathcal{Y}(x) = \left(\frac{-x}{D}\right)^{\alpha-1} \tag{3.46}$$

We see, $\mathcal{Y}(x)$ is positive for negative x and negative for positive x. It remains to show that $X \in L^p(P)$. For $x > 0$ this is trivially satisfied for $X = 0$. For $x < 0$, we again suppose that θ is bounded, we have $Z_{Q_0} \in L^q$ for all q.:

$$E(X_0^p(x)) = E(I^p(\mathcal{Y}(x)Z_{Q_0})) = (\mathcal{Y}(x))^{\frac{p}{\alpha-1}} E(Z_{Q_0}^{\frac{p}{\alpha-1}}) < \infty \tag{3.47}$$

The corresponding value function is then:

$$G(y) = -E\left(-\left(-(yZ_0)^{\frac{1}{\alpha-1}}\right)^{\alpha}\right) \cdot \frac{1}{\alpha} = -\frac{1}{\alpha} y^{\frac{\alpha}{\alpha-1}} E(Z_{Q_0}^{\frac{\alpha}{\alpha-1}}),\ y > 0.$$

$$V(x) = \begin{cases} G(\mathcal{Y}(x)) = -\frac{1}{\alpha}\left(\frac{x}{D}\right)^{\alpha} E\left(Z_{Q_0}^{\frac{\alpha}{\alpha-1}}\right) < 0, & x < 0 \\ 0 & x \geq 0 \end{cases} \tag{3.48}$$

We see, $G(\mathcal{Y}(x))$ is suboptimal to 0 for positive x.

Partial Sums of the Exponential Utility Function

In this section, we consider the even partial sums of the exponential function. We only take the even ones, since they are concave. Partial sums are interesting, because on the one hand they behave like iso-elastic functions with parameters bigger than one. The main interpretation is hedging the claim. On the other hand up to a certain point it is more like the exponential series. The even partial sums have a unique global maximum like the iso-elastic functions and so beyond this point, utility is decreasing again with increasing terminal wealth. So analogously, an investor with this utility function tries to hedge the maximum of this function. The difference is, this maximum is not zero, it tends to infinity, if n does. But as larger this maximum will get the similar the shape of the function will be to the exponential one, up to the maximum. As in the iso-elastic case, as long as we are below the maximum, we face a utility maximization problem of an increasing function. So we can expect, for every initial wealth there will be an n, such that the form of the solution of the partial sums will be quite similar and converge to the exponential one. In particular, the constraint will be fulfilled with equality. We do not consume. It is well known that the partial sums of the exponential series converge faster than $(1 + \frac{x}{n})^n$ to the exponential function. However, it is easier to show that the optimal terminal values converge for the second sequence, see section 5.2. In the complete case, we are also able to show this result for the faster converging sequence:

We define the even partial sums of the exponential function:

$$\tilde{u}_{2n}(x) = \sum_{k=0}^{2n} \frac{1}{k!} x^k \tag{3.49}$$

Since we analyzed $-e^{-x}$, we also switch to $u_{2n}(x) = -\tilde{u}_{2n}(-x)$. u_{2n} converges to $-e^{-x}$, is concave, but only increasing up to a certain x. But its derivative the odd partial sums

$$u'_{2n}(x) = u_{2n-1}(x) = \sum_{k=0}^{2n-1} \frac{1}{k!}(-x)^k$$

are strictly decreasing with $u_{2n-1}(\infty) = -\infty$, $u_{2n-1}(-\infty) = \infty$. Hence, the inverse of u'_{2n} exists and ranges over the whole real line. We call it I_n. We also have $I_n(-\infty) = \infty$, $I_n(\infty) = -\infty$. Moreover, $f = u_{2n-1}$ is strictly decreasing and differentiable with derivative $f' = u_{2n-2}(x)$, which is strictly negative on the real line. So we can apply the inverse function theorem: I_n is differentiable and

$$I'_n(y) = \frac{1}{f'(I_n(y))} = \frac{1}{u_{2n-2}(I_n(y))} < 0 \tag{3.50}$$

Before we go on, we define:

$$m_n := \max_x u_{2n}(x),\ M_n := \arg\max_x u_{2n}(x),\ n \in \mathbb{N}. \tag{3.51}$$

Further, it holds for every n and $x < 0$, that $e^x \leq \tilde{u}_{2n}(x)$. Hence

$$\max_x u_{2n-2}(x) = m_{n-1} < 0.$$

But this yields for each fixed n:

$$|I'_n(yZ_0)| = |\frac{1}{u_{2n-2}(I_n(yZ_0))}| \leq |\frac{1}{m_{n-1}}| \leq K_n \in \mathbb{R}^+ \tag{3.52}$$

By the dominated convergence theorem (or the corresponding corollary of exchanging of differentiation and integration, respectively), we have:

$$\mathcal{X}'_n(y) = E(Z_{Q_0}^2 I'_n(yZ_0)) < 0 \tag{3.53}$$

The last inequality follows by (3.50) and since $Z_{Q_0} > 0$. Thus, $\mathcal{X}$ is strictly decreasing and therefore also $\mathcal{Y}$. Property 1. follows. Further, since $|u_{2n-1}(x)|$ is bounded from below by $|x| + C_n$, where C_n a positive constant, hence $|I_n(y)| \leq |y| + C_n$. Consequently, we have for each fixed n:

$$\mathcal{X}_n(y) = E(Z_{Q_0} I_n(yZ_0)) \leq |y| E(Z_0^2) + \tilde{C}_n < \infty,\ y \in \mathbb{R}. \tag{3.54}$$

Since $\mathcal{X}_n(\infty) = -\infty$ and $\mathcal{X}_n(-\infty) = \infty$, property 2 and 3 are satisfied. Property 4. follows by the following estimation:

$$E(X_0^p(y)) = E(I_n^p(yZ_0)) \leq E(|yZ_0| + C)^p < \infty \tag{3.55}$$

Next, we have to find out when $\mathcal{Y}$ is positive and when consuming is necessary. We set $H_T = Z_0 e^{-\int_0^T r(s)ds}$ to give an example how the riskless return comes into account. In fact, we have:

Lemma 3.2.5. *$\mathcal{Y}_n(x)$ is positive if and only if the initial wealth is below the (discounted) maximum of the utility function,i.e. $x_n \equiv E(H_T)M_n > x$. $X_n(\mathcal{Y}(x))$ is the optimal solution. Otherwise, putting $x_n = E(H_T)M_n$ in the bank account and consuming the rest is optimal, the Lagrange multiplier is zero.*

Proof. The last assertion is obvious, since $E(u_{2n}(X_T)) = E(u_{2n}(M_n))$ and equal to the maximum. Since $\mathcal{Y}_n$ is strictly decreasing, we just have to show that $\mathcal{Y}_n(x_n) = 0$, or $\mathcal{X}_n(0) = x_n$, respectively. We have $u'_{2n}(I_n(0)) = u_{2n-1}(I_n(0)) = 0$ and u_{2n} is concave thus $I_n(0)$ is equal to the maximizer of u_{2n} and that is M_n. Hence, $\mathcal{X}_n(0) = E(H_T I_n(0)) = E(H_T M_n) = x_n$. □

Furthermore, the following holds:

Lemma 3.2.6. *For every initial wealth x there exists an n such that $x < x_n$. Consequently, $\mathcal{Y}_n(x) > 0$ and the optimal solution fulfills the constraints and is of the form: $X_n(\mathcal{Y}_n(x)) = I(\mathcal{Y}_n(x)H_T)$.*

Proof. $x_n = E(H_T I_n(0)) = E(H_T)m_n$ diverges to infinity, since m_n does. The five properties at the beginning of section 3.2.1 are satisfied. By Theorem 3.1.3, $X_n(\mathcal{Y}_n(x))$ is therefore optimal. □

Finally, we show, that the optimal solution in Lemma 3.2.6 converges to the optimal solution in the exponential utility case:

Theorem 3.2.7. *For all $x \in \mathbb{R}$, the optimal solution of the even partial sums converges pointwise to the optimal solution of the exponential utility function with parameter 1, i.e.*

$$X_n(\mathcal{Y}_n(x)) = \mathbf{1}_{\{x<x_n\}} I_n(\mathcal{Y}_n(x)H_T) + \mathbf{1}_{\{x_n \leq x\}} x_n \tag{3.56}$$

$$\xrightarrow{a.s.} I_{\exp}(\mathcal{Y}_{\exp}(x)H_T) = X_{\exp}(\mathcal{Y}_{\exp}(x)) \tag{3.57}$$

Proof. By Lemma 3.2.6, we just have to consider the first term of (3.56) and $\mathcal{Y}$ positive. So its inverse is defined on $(0, \infty)$. $u_{2n-1}(x)$ converges pointwise to e^{-x}, therefore its inverse $I_n(y)$ converges to $(e^{-\cdot})^{-1}(y) = -\log y$. It remains to show that $\mathcal{X}_n(y) \to \mathcal{X}_{\exp}(y)$, $y > 0$ because this yields the convergence of $\mathcal{Y}_n(x)$ to $\mathcal{Y}_{\exp}(x)$. Since all functions are locally Lipschitz continuous and converge almost surely, we obtain (3.57): Since there exist a

n_0 and $a, b \in (0, \infty)$ such that for all $n \geq n_0$, we have $\mathcal{Y}_n(x), \mathcal{Y}_{exp}(x) \in [a, b]$ and by the mean-value-theorem there exists an $\eta_n \in [a, b]$:

$$\begin{aligned} & |I_n(\mathcal{Y}_n(x)H_T) - I_{exp}(\mathcal{Y}_{exp}(x)H_T)| \\ \leq \; & |I_n(\mathcal{Y}_n(x)H_T) - I_n(\mathcal{Y}_{exp}(x)H_T)| \\ & +|I_n(\mathcal{Y}_{exp}(x)H_T) - I_{exp}(\mathcal{Y}_{exp}(x)H_T)| \\ = \; & |I_n'(\eta_n H_T)||\mathcal{Y}_n(x) - \mathcal{Y}_{exp}(x)| + |I_{exp}(\mathcal{Y}_n(x)H_T) - I_{exp}(\mathcal{Y}_{exp}(x)H_T)| \\ \stackrel{a.s.}{\to} \; & 0, \;\; \text{if } \mathcal{Y}_n(x) \to \mathcal{Y}_{exp}(x) \end{aligned}$$

since $\eta_n H_T(\omega) \in [aH_T(\omega), H_T(\omega)b] \subset [c, d],\ c, d \in (0, \infty)$ for $n \geq n_0$.

$I_n(yH_T)$ is integrable (see (3.54)) and for $y > 0$ and all n, it holds: $I_n(y) < |y|$. We have by dominated convergence:

$$\begin{aligned} \lim_{n\to\infty} \mathcal{X}_n(y) &= \lim_{n\to\infty} E(H_T I_n(yH_T)) \\ &= E(\lim_{n\to\infty} H_T I_n(yH_T)) = \mathcal{X}_{\text{exp}}(y). \end{aligned}$$

□

3.2.2 Incomplete Case and Examples

In this subsection, we switch to the incomplete case. This means that the set of the considered measures is not a singleton. Our main task is to solve problem (3.21). In the complete case this was fairly easy, since $\mathcal{M}_a^q$ was assumed to be a singleton. We introduce a method which can, but does not have to lead to a solution. First, we can try to solve (3.24). This seems to be rather challenging. (3.21) describes the dual problem. We solve it for Z in dependency of y and then try to solve (3.24) and establish a duality relationship:

Method in the Incomplete Case

One method to find a solution of the dual problem (3.21) is to fix y_0 and optimize $\phi(Z, y_0)$ among the measures, so that we end up with

$$Z(y_0) = \arg\min_Z \phi(Z, y_0). \tag{3.58}$$

As before, we have to find y_0 such that:

$$\mathcal{X}_{Z(y_0)}(y_0) = E(Z(y_0)I(Z(y_0)y_0)) = x \tag{3.59}$$

so the main task is to find out, if for every x such a y_0 exists and how we can determine it. For explicit utility functions, we might try to invert $\mathcal{X}_Z$,the optimal measure is independent of y_0. We can invert $\mathcal{X}$ without further problems and proceed as in the complete case.

Karatzas (1996) proves that (3.59) has a solution y_0 for every $x > 0$ in the case of utility functions of type I. The first step is to show, given certain assumptions, that the dual problem has a solution, i.e.

$$\forall y \in (0,\infty)\exists Z(y)\forall \tilde{Z} \in \mathcal{D} : \phi(\tilde{Z}, y) \geq \phi(Z(y), y) \tag{3.60}$$

see Theorem 5.5.1 ($\mathcal{D}$ corresponds to $\mathcal{M}$). But if (3.60) holds, then by Theorem 5.4.1 in Karatzas (1996) there exists an optimal superhedging strategy for every initial wealth of the form $\mathcal{X}_{Z(y)}(y), y \geq 0$. So as seen before, it remains to prove that for an arbitrary chosen $x > 0$, there exists a $\mathcal{Y}(x) \geq 0$ such that $x = \mathcal{X}_{Z(y)}(y),\ y = \mathcal{Y}(x)$ This and the stochastic duality, i.e. $V(x) = \inf_y[\phi(Z(y), y)]$, where the infimum is attained by $\mathcal{Y}(x)$, is the content of Theorem 5.4.2. We do not go further into detail since another setting is used.

Remark 3.2.2. The approach includes further considerations of market constraints. Portfolios have to be in a pre-given cone which models e.g. a prohibition on shortselling. Furthermore, the utility function also rewards consumption and admissible portfolios (gain process has to be bounded from below) are considered instead. Under suitable assumptions, the measure also induces the worst market completion, i.e. all value functions of different measure are bigger at the chosen initial wealth x. Conversely, if we find this completion, it will be the corresponding optimal measure, see Karatzas (1996), p.95/96.

The approach in Karatzas (1996) shows finding a solution of (3.59) is crucial, if we want to find a minimizer (Z_{opt}, y_0) of $\phi(Z, y)$. For a lot of utility functions, we can derive a solution directly (e.g. logarithmic or exponential utility functions). We suppose it is already given that for every x there exists a unique $\mathcal{Y}(x)$, which solves (3.59). We can build an inverse, i.e. the unique solution of (3.59) is defined as:

$$\mathcal{Y}(x) = \begin{cases} (\mathcal{X}_{Z(\cdot)})^{-1}(x) & (\mathcal{X}_{Z(\cdot)})^{-1}(x) > 0 \\ 0 & (\mathcal{X}_{Z(\cdot)})^{-1}(x) \leq 0 \end{cases} \tag{3.61}$$

Clearly, if $Z(y)$ is independent of y, we can proceed as in the complete case. $\mathcal{Y}$ can be easiliy derived by taking the inverse function. In all cases, the optimal solution of the primal problem (2.6) is given by:

$$X_0(x) = I(\mathcal{Y}(x)Z(\mathcal{Y}(x))),$$

provided property 3.2.1 is satisfies.

Duality in the Incomplete Case

We still suppose that (3.59) has a unique solution. When proving a duality result, we can proceed as in the complete case (3.32), if $Z(y)$ is independent of y. If this is not true, we cannot get rid of $Z(y_1)$ in the first equality (see below). Nevertheless, we can still use the same arguments:

$$\begin{aligned}
\phi(Z(y_1), y_1) &= E(\check{U}(Z(y_1), y_1)) + xy_1 \\
&= E(\max_X (U(X) - Z(y_1) \cdot y_1 X)) + xy_1 \\
&= E(U(X_0(y_1)) - Z(y_1) \cdot y_1 X_0(y_1)) + xy_1 \\
&\geq E(U(X_0(\mathcal{Y}(x))) - Z(y_1) \cdot y_1 X_0(\mathcal{Y}(x))) + xy_1 \\
&= E(U(X_0(\mathcal{Y}(x)))) + y_1(x - E(Z(y_1) X_0(\mathcal{Y}(x)))) \\
&\geq E(U(X_0(\mathcal{Y}(x))) \\
&= V(x)
\end{aligned} \tag{3.62}$$

where $Z(y_1)$ as in (3.58) and $X_0(y_1) = I(Z(y_1) \cdot y_1)$. The last inequality in (3.62) holds since for the optimal solution X_0, we have that $E(ZX_0) \leq x$ for all Z. The last equality can be proven analogously to the complete case:

$$y_1 E(Z(\mathcal{Y}(x)) X_0(\mathcal{Y}_{Z(y_1)}(x))) = y_1 E(Z(\mathcal{Y}(x)) I(Z(\mathcal{Y}(x)) \mathcal{Y}(x))) \tag{3.63}$$
$$= y_1 \mathcal{X}_{Z(\mathcal{Y}(x))}(\mathcal{Y}(x))) = y_1 x \tag{3.64}$$

We obtain equality in (3.62), if $y_1 = \mathcal{Y}(x)$. We have:

$$\min_{y_1 \geq 0} \phi(Z(y_1), y_1) = V(x) = \max_{X: \forall Z \, E(ZX) \leq x} E(U(X)) \tag{3.65}$$

In some sense this result is rather obvious. Since, if we have a solution of (3.59), we have found a saddlepoint, but by Proposition 3.2.1 this was equivalent to the minimax-theorem. In principle, this is the idea of Kramkov and Schachermayer (1999), imitating a minimax theorem and then deriving a saddlepoint. However, if the utility function is given, it is sometimes easier to find the saddlepoint directly. In the next section, we consider exponential utility functions in an incomplete market, we use the approach described above to get a saddlepoint.

Exponential Utility Function

In the sequel, we apply the stochastic version of Theorem 3.1.3, introduced in the previous subsections finishing with equations (3.59) and (3.58) for the incomplete case. This is, in fact, a similar approach to Delbaen et al (2002) and Kabanov and Stricker (2002) in a general semi-martingale model using different classes of strategies. They explicitly use the form of the minimal entropy measure, which is the solution of the dual problem. Kabanov and Stricker (2002) show that this minimizer with respect to the densities of $\mathbb{P}_f$ has the form

$$Z_{min} = e^{\min_{Q \in \mathbb{P}_f} H(Q|P) - \eta}, \tag{3.66}$$

where

$$\eta \in \mathcal{C} = \{\eta \in L^0 : \forall Z \in \mathcal{M}_{a,Z}\ E(Z|\eta|) < \infty,\ EZ\eta \leq 0\}.$$

Using this they derive that η is the optimal solution of the primal problem. We proceed the other way around. We use the above approach and end up with an optimal solution of the primal problem as a function of the minimal entropy measure. Calculating the inverse exactly gives us (3.66), where $\eta = X_0(\mathcal{Y}(0))$ the optimal solution of the primal problem. Both approaches give a duality between the solutions. We apply the approach above and derive a duality result which corresponds to the expected duality (3.27). The other approach is explained in section 3.2.2 (Utility Functions of Type II). Finally, we explain how to include contingent claims in the problem. As mentioned in remark 3.2.1, in principal this is just a change of measure under suitable conditions, e.g. that this change preserves certain properties.

We start formulating the main assumptions of the general model:

Assumption 3.2.3. *There exists a measure $Q \in \mathcal{M}_a^q$ with finite relative entropy, i.e. $H(Q|P) < \infty$. Equivalently $\mathbb{P}_f(P) = \{Q \in M_a^q : H(Q|P) < \infty)\} \neq \emptyset$.*

Assumption 3.2.4. $\mathcal{M}_e^q \cap \mathbb{P}_f(P) \neq \emptyset$

Using these two assumptions Frittelli (2002) (Theorem 2.1, 2.2) prove the following theorem:

Theorem 3.2.8. *If assumption 3.2.3 is satisfied then there exists a unique martingale measure Q_{min} with density Z_{min} that minimizes $H(Q|P)$ over all $Q \in P_f(P)$. If in addition assumption 3.2.4 is fulfilled, then $Q_{min} \in \mathcal{M}_e^q$, i.e. Q_{min} is equivalent to P.*

Q_{min} is known as the minimal entropy measure, see section 4.1.2.

Remark 3.2.3. In the Brownian Model, we can work with the class of martingale measures with bounded θ, see Theorem 2.2.3. The suprema over both sets are equal. So we only have to ensure that the minimizer is actually attained in this class. By proposition 3.2. in Kabanov and Stricker (2002) and Bellini and Frittelli and Delbaen et al (2002), we only have that this minimum exists within the class of equivalent martingale measures.

The setting of Rouge and El Karoui (2000) is slightly different, but transferable and so also the result: The minimal entropy measure is generated by a bounded θ.

We state one more assumption to include contingent claims:

Assumption 3.2.5. *ξ, symbolizing a contingent claim, is bounded from below and*

$$\exists \epsilon > 0 : \ E(e^{(\alpha+\epsilon)\xi}) < \infty.$$

We will need assumption 3.2.5 to show that $\mathbb{P}_f(P) = \mathbb{P}_f(P_\xi)$, where P_ξ is defined by:

$$\frac{dP_\xi}{dP} = c_\xi e^{\alpha\xi}, c_\xi = (E(e^{\alpha\xi}))^{-1} \tag{3.67}$$

In a Brownian model assumption 3.2.3 and assumption 3.2.4 are already satisfied, if we have one martingale measure Q with a $Q-$ square-integrable θ. This is the content of the following lemma:

Lemma 3.2.9. *In the Brownian Model with bounded coefficients assumption 3.2.3 and 3.2.4 are automatically satisfied.*

Proof. We choose $\bar{\theta} = \sigma'(\sigma\sigma')^{-1}\mu$. By Novikov's condition, this generates an equivalent martingale measure:

$$\frac{dQ}{dP} = Z = \mathcal{E}_T(\int \bar{\theta} dW).$$

We only have to show that this measure has finite entropy (by Frittelli (2002), Theorem 2.2 see above). By Girsanov's theorem we have that $\tilde{W}_t = W_t + \int_0^t \bar{\theta}(s)ds$ is a Brownian motion under Q. Using this, we obtain

$$\begin{aligned} H(Q|P) &= E(Z \log Z) = E(Z(-\int_0^T \theta(\bar{t})dW_t - \frac{1}{2}\int_0^T \|\bar{\theta}(t)\|^2(t)dt)) \\ &= E_Q(-\int_0^T \bar{\theta}(t)d\tilde{W}_t + \frac{1}{2}\int_0^T \|\bar{\theta}(t)\|^2(t)dt) \\ &= 0 + E_Q(\frac{1}{2}\int_0^T \|\bar{\theta}(t)\|^2(t)dt) = (*) \end{aligned}$$

$E_Q(-\int_0^T \bar{\theta}(t)d\tilde{W}_t) = 0$, since $E_Q(\int_0^T \|\bar{\theta}_t\|^2 dt) < \infty$.

Since $\bar{\theta}$ is already bounded:

$$(*) \leq 0 + \frac{1}{2}TME(Z) = \frac{1}{2}TM < \infty$$

where $M = \sup_{t,\omega} \|\bar{\theta}\|^2(\omega, t) < \infty$ □

By Lemma 3.2.9 it suffices to maximize over $\mathbb{R}^+ \times \mathbb{P}_f^Z(P)$. The index Z symbolizes that we mean the corresponding densities.

Next, we derive the optimal X_0. By (3.19) we obtain:

$$X_0(Z_1 y_1) = I(Z_1 y_1) = -\log(Z_1 y_1) \tag{3.68}$$

where $(Z_1 \cdot y_1)$ is the minimizer of

$$\begin{aligned} \phi(y_1, Z_1) &= E(U(I(Z_1 \cdot y_1)) - Z_1 \cdot y_1 I(Z_1 \cdot y_1)) + xy_1 \\ &= E(-\exp -(\log(Z_1 \cdot y_1)) + Z_1 \cdot y_1 \log(Z_1 \cdot y_1)) + xy_1 \\ &= -y_1 + xy_1 + y_1 \log y_1 + y_1 E(Z_1 \log Z_1) \\ &= -y_1 + xy_1 + y_1 \log y_1 + y_1 H(Q_1|P) \end{aligned}$$

with $Z_1 = \frac{dQ_1}{dP}$. We have $y_1 \geq 0$, so explained at the beginning of section 3.2.2 (see (3.58)), we start deriving $Z(y_1)$:

$$Z(y_1) = \arg\min_Z \phi(Z, y_1)$$

Clearly, $Z(y_1)$ is equal to the minimal entropy measure $Z_m = \arg\min_Q H(Q|P)$ and independent of y_1, therefore also from the initial wealth x. It remains to prove that Z_m exists, see Frittelli (2002) for details. To determine y_1, we apply the result from equation (3.59, $\mathcal{X}_{Z(y_1)}(y_1) = x$):

$$\begin{aligned} \mathcal{X}_{Z(y_1)}(y_1) &= \mathcal{X}_{Z_m}(y_1) = E(Z_m I(Z_m y_1)) \\ &= E(Z_m - \log(Z_n y_1)) = -E(Z_m \log Z_m) - \log y_1 \\ &= -H(Q_m|P) - \log y_1 \end{aligned}$$

We calculate the inverse of $\mathcal{X}$ and finally obtain the solution:

$$\begin{aligned} \mathcal{Y}(x) &= \exp\{-H(Q_m|P) + x\} \\ X_0(x) &= X_0(\mathcal{Y}(x)) = I(Z_m \mathcal{Y}(x)) \\ &= -\log Z_m + H(Q_m|P) + x \end{aligned} \tag{3.69}$$

We see the optimal strategy to reach X_0 does not depend on the initial wealth. Hence, from (3.66) we get:

$$Z_m = \exp(-(X_0(0)) + H(Q_m|P)) = Z_{min}.$$

Since, $EZ_m = 1$, we have:

$$Z_m = \frac{\exp(-(X_0(0))}{E(\exp(-(X_0(0)))} = \frac{u'(X_0(0)}{E(u'(X_0(0))}. \tag{3.70}$$

Further, we see that $Z_{min} > 0$. The minimal entropy measure is therefore an equivalent measure.

By plugging the optimal solution into $\sup_{X \in \mathcal{O}} E(-e^{-X-x})$, we obtain a duality under an arbitrary probability measure P with $\mathbb{P}_f \cap \mathbb{P}_e \neq \emptyset$:

$$\sup_{X \in \mathcal{O}} E(-e^{-X-x}) = -e^{-x-\min_Q H(Q|P)} \tag{3.71}$$

Note, we still have to prove that $X_0 \in \mathcal{O}$. Then property 3.2.1 is satisfied. $EU(X_0) < \infty$ is trivial as seen in the complete case. By (3.69) $X \in L^p(P)$ boils down to $E(\log^p Z_m) < \infty$. If the corresponding parameter in the Girsanov functional θ^m is bounded, we are done, as shown before:

$$E \log^p Z_m = E(-\int_0^T \theta_s^m dW_s - \frac{1}{2}\int_0^T (\theta_s^m)^2 ds)^p \tag{3.72}$$

$$\leq C_1 E(W_T - C_2)^p \tag{3.73}$$

The minimal entropy measure is equal to the minimal martingale measure, if the coefficients are deterministic (see sections 4.1.1 and 4.1.2). So in this case $\theta_m = \bar{\theta}$, which is bounded, if the coefficients σ and μ are bounded. Even more generally for $p \in [1, \infty)$: From theorem 2.2.6, we know that the minimal entropy measure is of the form $\mathcal{E}_T(M^{Q_{min}})$. For $p \in (1, 2]$ and if $E((\langle M^{Q_{min}} \rangle)^2) < \infty$, we can use the isometry property of the stochastic integral and by Young's inequality, we have:

$$\begin{aligned} E \log^2 Z_{min} &= E(M^{Q_{min}} - \frac{1}{2}[M^{Q_{min}}])^2 \\ &= E((M^{Q_{min}})^2 - 2M^{Q_{min}} \frac{1}{2}[M^{Q_{min}}] \\ &\quad + (\frac{1}{2}[M^{Q_{min}}])^2) \\ &\leq E(2(M^{Q_{min}})^2 + 2(\frac{1}{2}[M^{Q_{min}}])^2 \\ &\leq 2E([M^{Q_{min}}] + 2(\frac{1}{2}[M^{Q_{min}}])^2) < \infty \end{aligned}$$

Let $E((M^{Q_{min}})^{2p}_T) < \infty$, then:

$$\begin{aligned} E \log^p Z_{min} &= E(M_T^{Q_{min}} - \frac{1}{2}[M^{Q_{min}}]_T)^p \\ &\leq 2^{p-1} E((M_T^{Q_{min}})^p + (\frac{1}{2})^p [M^{Q_{min}}]^p_T) \\ &\leq C(p) \cdot E((M_T^{Q_{min}})^{2p}) < \infty \end{aligned} \tag{3.74}$$

by the inequalities Burkholder-Davis-Gundy and Doob and $C(p)$ a positive constant dependent on p. Z_{min} is the minimal entropy measure and so $E(M_T^{Q_{min}})^{2p} < \infty$, since $(M_t^{Q_{min}})$ is bounded under the assumptions of Theorem 4.1.10.

Finally, we want to treat the case, where we face a random nonzero contingent claim:

$$\sup_{X \in \mathcal{O}} E(-\exp\{-\alpha(x + X - \xi)\}) \tag{3.75}$$

We would like to show a duality result of the following form:

$$\sup_{X \in \mathcal{O}} E(-e^{-\alpha(x+X-\xi)}) = -\exp\{\alpha \sup_{Q \in \mathbb{P}_f} (E_Q(\xi) - x - \frac{1}{\alpha} H(Q|P))\} \tag{3.76}$$

We already solved this problem for $\xi \equiv 0$ (problem (3.71)). The goal is now to reduce problem (3.76) to (3.71). This was fully developed in Delbaen et al (2002). We ignore α, which is just a matter of standardization. Clearly, (3.71) also holds for P_ξ if $\mathbb{P}_{\xi_f} \cap \mathbb{P}_{\xi_e} \neq \emptyset$. It is rather inconvenient to check this assumption for every claim. However, we have the following lemma:

Lemma 3.2.10. *If ξ is bounded from below and $E(e^{(\alpha+\epsilon)\xi}) < \infty$ for some $\epsilon > 0$, then $\mathbb{P}_{\xi f} = \mathbb{P}_f$ and consequently also the intersection with $\mathbb{P}_e$ (P and P_ξ are equivalent.)*

Proof. Lemma 3.5. in Delbaen et al (2002) states that for a ξ bounded from below or $\xi \in L^1$, the following holds:

$$E_Q(\gamma\xi) \leq H(Q|P) + \frac{1}{e}E(e^{\gamma\xi}), Q \ll P, \gamma > 0 \tag{3.77}$$

Since $E_{P_\xi}(e^{\epsilon\xi}) = CE(e^{(\alpha+\epsilon)\xi}) < \infty$ by assumption for a positive constant C, we obtain by Lemma 3.5. that $E_Q(\epsilon\xi) < \infty$ and therefore $E_Q(\alpha\xi) < \infty$. Further, it holds that

$$H(Q|P_\xi) + \log c_\xi + E_Q(\alpha\xi) = H(Q|P) \tag{3.78}$$

hence, $H(Q|P)$ is finite if and only if $H(Q|P_\xi)$ is finite. Thus, $\mathbb{P}_{\xi f} = \mathbb{P}_f$. □

Next, we start transforming (3.76). First, recognize that x does not influence the optimal $X(0)$, so we just divide both sides of (3.76) by e^{-x}. The remaining assertions are content of the next proposition, using the arguments in Delbaen et al (2002):

Proposition 3.2.11. *Let $\alpha = 1$, then (3.76) is equivalent to:*

$$\sup_{X\in\mathcal{O}} E_{P_\xi}(-e^{-X}) = -e^{-\inf_Q H(Q|P_\xi)},$$

i.e.:

$$\begin{aligned}\sup_{X\in\mathcal{O}} E(-\exp\{(x + X - \xi)\}) &= -\exp\{\sup_{Q\in\mathbb{P}_f}(E_Q(\xi) - x - H(Q|P))\}\\ &= \sup_{X\in\mathcal{O}} E_{P_\xi}(-e^{-X})\\ &= -e^{-\inf_Q H(Q|P_\xi)}.\end{aligned}$$

Proof. First, we transform the LHS of (3.76)

$$\begin{aligned}-E(e^{-\alpha X+\alpha\xi}) &= -E_{P_\xi}(\frac{dP}{dP_\xi}e^{-\alpha\xi}e^{-\alpha X})\\ &= -E_{P_\xi}(c_\xi^{-1}e^{-\alpha\xi}e^{\alpha\xi}e^{-\alpha X})\\ &= -E_{P_\xi}(c_\xi^{-1}e^{-\alpha X})\end{aligned}$$

and the RHS:

$$\begin{aligned}&-\exp\{\alpha \sup_{Q\in\mathbb{P}_f}(E_Q(\alpha\xi) - \frac{1}{\alpha}H(Q|P))\}\\ =& -\exp\{-\alpha \inf_{Q\in\mathbb{P}_f}(-E_Q(\alpha\xi) + \frac{1}{\alpha}H(Q|P))\}\\ =& -\exp\{-\alpha \inf_{Q\in\mathbb{P}_f}(H(Q|P_\xi) + \log c_\xi^{-1})\}\end{aligned}$$

by (3.78).

$$
\begin{aligned}
& -\exp\{-\alpha \inf_{Q\in\mathbb{P}_f}(H(Q|P_\xi)+\log c_\xi^{-1})\} \\
= & -\exp\{-\alpha(\inf_{Q\in\mathbb{P}_f}(H(Q|P_\xi))+\log c_\xi^{-1})\} \\
= & -c_\xi^{-1}\exp\{-\alpha \inf_{Q\in\mathbb{P}_f}(H(Q|P_\xi))\}
\end{aligned}
$$

□

Hence, proposition 3.2.11 and (3.71) for $P = P_\xi$ yield a duality result for claims satisfying 3.2.5. From (3.69), we obtain the optimal solution:

$$
X_\xi(x) = I(Z_m\mathcal{Y}(x)) = -\log Z_m^\xi + H(Q_m^\xi|P) + x \tag{3.79}
$$

where $Q_m^\xi = \arg\min_Q H(Q|P_\xi)$ and Z_m^ξ the corresponding density.

Exponential Utility and Microeconomic Equilibrium

Up to now, we always required that the budget constraint $(E_Q(X) \le x)$ is satisfied under every measure. Since we did not know which measure we faced. Now, we assume that we have a finite set of investors, say $\mathcal{I}$. Each investor a has an initial endowment of assets which has a payoff of W_a in period T. He wants to restructure his portfolio such that the corresponding payoff X_a maximizes his expected utility with parameter α_a. However, someone must be willing to sell (buy) the products you want to buy (sell), every investor wants to do an optimal reconstruction of his portfolio. This is regulated buy the price. The sum of the new payoffs has to be equal to the sum of the old payoffs. We would like to know under which price measure Z this is possible, i.e. when we live in an equilibrium. That means:

Definition 3.2.3. A price density Z_e and a set of payoffs $(X_a^e)_{a\in\mathcal{I}} \in L^p$ with $\sum_{a\in\mathcal{I}} X_a = \sum_{a\in\mathcal{I}} W_a \equiv W$ forms an Arrow-Debreu equilibrium, if X_a^e solves the utility maximization problem of investor a with respect to Z_e.

This concept is introduced for the discrete case in Föllmer and Schied (2002). We already know the solution of the individual optimization problem. It is, see e.g. (3.69):

$$
X_0^a(x) = -\frac{1}{\alpha_a}\log Z_e + \frac{1}{\alpha_a}H(Z_e|P) + x_a \tag{3.80}
$$

If these set $(X_0^a)_a$ shall be part of an equilibrium, then $W = \sum X_0^a$:

$$
W = -\frac{1}{\alpha}\log Z_e + \frac{1}{\alpha}H(Z_e|P) + \sum_a x_a \tag{3.81}
$$

where $\frac{1}{\alpha} = \sum_a \frac{1}{\alpha_a}$. Hence, Z^e must have the form:

$$Z_e = \frac{e^{-\alpha W}}{E(e^{-\alpha W})} \tag{3.82}$$

which looks like the minimal entropy measure. But W cannot be changed, it is not the solution of a primal problem. We have to prove that Z_e exists, we assume:

Assumption 3.2.6. *We have*

$$E(|W_a|e^{-\alpha W}) < \infty, \ a \in \mathcal{J}.$$

It is satisfied if e.g. W_a is bounded from below. Since $W_a \in L^p \subset L^1$, assumption 3.2.6 is automatically satisfied in our case, provided the over all wealth W is positive:

$$E(|W_a|e^{-\alpha W}) \leq E(|W_a|) < \infty, \ a \in \mathcal{I}.$$

Define $\frac{dQ_e}{dP} = Z_e$, then we have

$$H(Q_e|P) = -\alpha E_e(W) - \log E(e^{-\alpha W}) < \infty$$

and X_0^a takes the form:

$$X_a^* = x_a + \frac{\alpha}{\alpha_a}(W - E_e(W)) \tag{3.83}$$

since $\sum_a x_a = E_e(W)$, we have $\sum_a X_a^* = W$ and therefore an equilibrium.

Utility Functions of Type II

In this section, we treat utility functions of type II. $-e^{-x}$ is contained in this class. The wealth process can become negative. Schachermayer (2001) completely solved this optimization problem. However, he used another set of trading strategies, an approximation of wealth processes generated by admissible strategies, i.e. processes with a finite credit line. This class is to narrow to find a maximizer, so an approximation is needed. In the exponential problem we show under certain assumption the optimal wealth process is equal to our process generated by strategies in $\mathcal{A}^p$. So in this sense, when using strategies in $\mathcal{A}^p$, we also obtain an optimal strategy that (approximately) only makes use of a finite credit line.

We start with the definition of admissible strategies:

Definition 3.2.4. An admissible strategy N is a predictable, S-integrable process with $\int_0^t N_u dS_u$ uniformly bounded from below:

$$P(\exists c_N \forall t \in [0,T] : \int_0^t N_u dS_u \geq c_N) = 1$$

Let X_b denote the set of wealth processes generated by admissible strategies:

$$X_b = \{Y | Y = (Y_t)_{t\in[0,T]},\ Y_t = \int_0^t N_u dS_u + x,\ \text{N admissible}\}$$

The interpretation is clear. The investor is only equipped with a finite credit line. Doubling strategies are excluded. This class turns out to be too small to identify a maximum. So Schachermayer (2001) chooses a set of random variables closest to the final values of processes in X_b. The author takes an approximation of random variables dominated by final values of admissible processes. Afterwards, he shows that the optimal random variable is indeed a final value of a wealth process using a predictable S-integrable strategy! Note, our class also excludes doubling strategies, since our wealth processes are already martingales, see Lemma 1.2.5 in the Brownian case. This can be extended by Burkholder-Davis-Gundy-inequality.

The approximation is defined as follows:

Definition 3.2.5. We define

$$C_u^b = \{G_T \in L^0(\Omega, \mathcal{F}_T, P) |\ G_T \le Y_T \text{ for some } Y \in X_b,\ E|U(G_T)| < \infty\}$$

and

$$C_u = \{F_T \in L^0(\Omega, \mathcal{F}_T, P; \mathbb{R} \cup \infty) |\ U(F_T) \in \overline{\{U(G_T)|G_T \in C_u^b\}}\},$$

where $L^0(\Omega, \mathcal{F}_T, P)$ the space of all $\mathcal{F}_T$- measurable functions endowed with topology of convergence in probability.

We again base this section on the semimartingale model defined in the beginning, but with admissible trading strategies. We also assume that assumption 1.2.3 holds true. Next, we introduce the class of utility functions of type II:

Definition 3.2.6. A utility function of type II is a finitely-valued, continuous differentiable, increasing, strictly concave function $U : \mathbb{R} \to \mathbb{R}$ with the following properties:

1. $U'(\infty) := \lim_{x\to\infty} U'(x) = 0$
2. $U'(-\infty) := \lim_{x\to-\infty} U'(x) = \infty$

In this section, we only treat utility functions of this type. We give an example:

Example 3.2.2. The exponential function $\tilde{U}(x) = 1 - e^{-x}$ or $U(x) = e^{-x}$ are utility functions of type II:

$$U'(x) = e^{-x} \Rightarrow U'(\infty) = \lim_{x\to\infty} e^{-x} = 0,\ U'(-\infty) = \lim_{x\to-\infty} e^{-x} = \infty$$

■

Schachermayer (2001) considers the following problem:

$$\sup_{F_T \in C_u(x)} E(U(F_T)) = \sup_{G_T \in C_u^b(x)} E(U(G_T)) =: u(x) \tag{3.84}$$

A further assumption is needed:

Assumption 3.2.7. *The value function $u(x)$ satisfies:*

$$u(x) < U(\infty) := \lim_{x\to\infty} U(x)$$

This is again satisfied in the case of an the exponential utility function. Finally, we need the following concept:

Definition 3.2.7. A utility function of type II has reasonable asymptotic elasticity, if the following conditions hold:

1. $AE_{-\infty}(U) = \liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1$
2. $AE_{\infty}(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$

We continue example 3.2.2:

Example 3.2.3. *(example 3.2.2 continued)*
$U(x) = -e^{-x}$ has reasonable asymptotic elasticity:

$$U(x) = -e^{-x} \Rightarrow \frac{xU'(x)}{U(x)} = \frac{xe^{-x}}{-e^{-x}} = -x \tag{3.85}$$

Hence,

$$\begin{aligned} AE_{-\infty}(U) &= \liminf_{x\to-\infty} -x = \infty > 1 \\ AE_{+\infty}(U) &= \liminf_{x\to\infty} -x = -\infty < 1 \end{aligned}$$

■

See Schachermayer (2001), p. 5, for further examples and interpretations of this notion.

The primal problem in (3.84) corresponds to the following dual problem:

$$v(y) = \inf_{Q\in\mathcal{M}_e} E(\check{U}(y\frac{dQ}{dP})) \tag{3.86}$$

where $\check{U}$ is the conjugate function to U, i.e. $\check{U}(y) = \sup_{x\in\mathbb{R}}(U(x)-xy),\ y > 0$ as before. Since U is of type II, $\check{U}$ is a smooth, convex function satisfying:

$$\check{U}(0) = U(\infty),\ \check{U}(\infty) = \infty,\ \check{U}'(0) = -\infty,\ \check{U}'(\infty) = \infty, \tag{3.87}$$

and

$$\check{U}(y) = U(I(y)) - yI(y),\ U'(\cdot) = (-\check{U}')^{-1}(\cdot) := I(\cdot). \tag{3.88}$$

Note, the dual problem is an optimization over the class of all equivalent martingale measures. Whereas for utility functions of type I -in a very general setting- the domain ($\mathcal{M}_a$) is too small (see Kramkov and Schachermayer (1999)), $\mathcal{M}_a$ is big enough for utility functions of type II to attain the minimum. The minimum is a probability measure.

We now state the theorem from Schachermayer (2001), p.15:

Theorem 3.2.12. *In the general Semimartingale Model from section 1.2.1 including assumption 1.2.3, let U a utility function of type II with reasonable asymptotic elasticity satisfying assumption 3.2.7. Then,*

1. *The value functions $u(v)$ are finitely valued, strictly concave (convex), differentiable on $\mathbb{R}$ ($\mathbb{R}^+$). u and v are conjugate, i.e.:*

$$\begin{aligned} v(y) &= \sup_{x\in\mathbb{R}}(u(x) - xy),\ y > 0 \\ u(x) &= \sup_{y\in\mathbb{R}^+}(v(y) + xy),\ x \in \mathbb{R} \end{aligned}$$

   and satisfy:

$$u'(\infty) = 0,\ u'(-\infty) = \infty,\ v'(0) = -\infty,\ v'(\infty) = \infty$$

   The value function u has reasonable asymptotic elasticity.

2. *For $y > 0$ the optimal solution $\hat{Q}(y) \in \mathcal{M}_a$ to the dual problem (3.86) uniquely exists.*

3. *For $x \in \mathbb{R}$ the optimal solution $\hat{F}_T(x) \in C_U(x)$ to the primal problem (3.84) uniquely exists and is given by*

$$\hat{F}_T(x) = I(y\frac{d\hat{Q}(y)}{dP}), u'(x) = y \tag{3.89}$$

4. *If $\hat{Q}(y) \in \mathcal{M}_e$ and $x = -v'(y)$, then*

$$\hat{F}_T(x) = Y_T(x),\ Y_t(x) = x + \int_0^t N_u dS_u$$

   where N predictable and $S-$integrable and Y is a uniformly integrable martingale under $\hat{Q}(y)$.

5. *The following formulae hold true:*

$$\begin{aligned} v'(y) &= E(\frac{d\hat{Q}(y)}{dP}\check{U}'(y\frac{d\hat{Q}(y)}{dP})) \qquad (3.90) \\ xu'(x) &= E(Y_T(x)U'(Y_T(x))) \qquad (3.91) \end{aligned}$$

For a proof, we refer to Schachermayer (2001). It mainly relies on a previous paper of its author Kramkov and Schachermayer (1999). An optimality result is given for utility functions of type I. It imitates the minimax principle given in Theorem 3.2.1. This leads to a saddlepoint and to a solution of the primal problem. Utility functions of type I are defined on $(0,\infty)$. By a simple transformation the result is also valid for functions defined on $[-n,\infty)$. The device is now to approximate the utility function of type II by utility functions of type I and use the theorem from Kramkov and Schachermayer (1999). Interestingly, the minimizer of the dual problem in the present case turns out to be in $\mathcal{M}_a$. In Kramkov and Schachermayer (1999), the dual minimizer is only in the following bigger class:

$$\mathcal{Z}(y) = \{Z \geq 0 | Z_0 = y,\ ZY = (Z_t Y_t)_t \text{ is a supermartingale for all } Y \in \mathrm{Y}(1)\}$$

where

$$\mathrm{Y}(x) = \{Y \geq 0 | Y_t = x + \int_0^t N_u dS_u,\ \text{N predictable and } S-\text{integrable}\}$$

Note, the class $\mathcal{Z}(y)$ contains processes. $Z \neq Z_T$ in this situation.

Let us compare this result with our previous result. The dual problem (3.86) coincides with the one, we proposed before. So item 2 also holds in our case. Further, the optimal solution $F_T(x)$ has exactly the same form as in (3.62):

$$\begin{aligned}
\hat{F}_T(x) &= I(y\frac{d\hat{Q}(y)}{dP}),\ u'(x) = y \text{ or } v'(y) = -x \\
& \quad v'(y) = E(\frac{d\hat{Q}(y)}{dP}\breve{U}'(y\frac{d\hat{Q}(y)}{dP})) \\
&\Leftrightarrow -E(\frac{d\hat{Q}(y)}{dP}I(y\frac{d\hat{Q}(y)}{dP})) = -x \\
&\Leftrightarrow \mathcal{X}_{Q(y)}(y) = E(\frac{d\hat{Q}(y)}{dP}I(y\frac{d\hat{Q}(y)}{dP})) = x \\
&\triangleq \mathcal{X}_{Z(y_0)}(y_0) = E(Z(y_0)I(y_0 Z(y_0))) = x \text{ in (3.59)}
\end{aligned}$$

Note, we can replace $u'(x) = y$ by $v'(y) = x$ since both are conjugate. The dual problems (3.21) and (3.86) coincide. $\mathcal{M}_e^q$ is dense in $\mathcal{M}_a$ and therefore the solution of both problems are equal, if they exist, (see e.g. in the exponential problem if $\mathcal{M}_e \neq \emptyset$ then the minimal entropy measure is also an equivalent measure). So if $I(\mathcal{Y}(x)\frac{d\hat{Q}(y)}{dP})$ has the right integrability, both primal problems have a solution and are equal, see e.g. the integrability discussion for the exponential case (3.74). This finds support in the following consideration. Suppose $\hat{F}_T(x) \in L^p$. Item 4 of Theorem 3.2.12 implies that there is an $S-$integrable predictable process N such that, for some $Q \in \mathcal{M}_e^q$ the process $(\int^t N_u dS_u)_t$ is a uniformly integrable $Q-$martingale converging to $\hat{F}_T(x)$ in the norm of $L^1(Q)$. By Theorem 2.2.4 this is equivalent to $\hat{F}_T(x) \in \overline{\mathrm{V}^p} = \overline{\mathcal{G}^p}$. So in the case of an exponential utility function, we have under the assumptions of Theorem 4.1.10 below (which implies that $\hat{F}_T(x) \in L^p$, see (3.74)) that:

$$\begin{aligned} F_T(x) &= I(\mathcal{Y}(x)\frac{d\hat{Q}(\mathcal{Y}(x))}{dP}) \in \mathcal{G}^p(x) \\ &= x + \int_0^T N_u dS_u,\ N \in \mathcal{A}^p \end{aligned}$$

This proves again the existence of an optimal solution of the exponential problem in the setting with H^p- integrable strategies, i.e. $N \in \mathcal{A}^p$.

Our approach has the disadvantage that we have to prove certain integrability properties, but we can use the nice properties of a reflexive Banach space. The discussion of different trading strategies can be extended. So next, we introduce a class of trading strategies which also comes close to the concept of admissible strategies. For utility functions of type II, Schachermayer (2002) proves similar results as above for these strategies. The disadvantage is that $\int NdS$ has to be a $Q-$supermartingale for every $Q \in \mathcal{M}_a$. This problem vanishes when switching to the exponential utility function. Similar results on the exponential utility function are obtained in Delbaen et al (2002) and Kabanov and Stricker (2002).

As seen for example in the exponential case, the unique minimizer of the dual problem exists and it is given by the minimal entropy measure. The existence can be proven in a lot of other cases, see Bellini and Fritelli (2002). So Schachermayer (2002) just assumes the minimizer $\hat{Q}(y)$ of the dual problem uniquely exists and defines the following classes of trading strategies:

Definition 3.2.8. Let $x \in \mathbb{R}$:

1. A predictable, $S-$integrable process N is in $\mathbf{H}_1$, if $U(x+\int_0^T N_u dS_u) \in L^1(P)$ and $(\int_0^t N_u dS_u)_t$ is a supermaringale under $\hat{Q}(y)$, where $u'(x) = y$.
2. A predictable, $S-$integrable process N is in $\mathbf{H}_2$, if $U(x+\int_0^T N_u dS_u) \in L^1(P)$ and $(\int_0^t N_u dS_u)_t$ is a supermaringale under every measure $Q \in \mathcal{M}_a$ with $E(\check{U}(\frac{dQ}{dP})) < \infty$
3. A predictable, $S-$integrable process N is in $\mathbf{H}_3$, if $U(x+\int_0^T N_u dS_u) \in L^1(P)$ and $(\int_0^t N_u dS_u)_t$ is a supermaringale under every measure $Q \in \mathcal{M}_a$ with $E(\check{U}(\frac{dQ}{dP})) < \infty$, and there exists a sequence $(N^n)_{n\in\mathbb{N}}$ of admissible trading strategies such that:

$$\lim_{n\to\infty} E(U(x + \int_0^T N_u dS_u \wedge \int_0^T N_u^n dS_u)) = E(U(x + \int_0^T N_u dS_u))$$

4. $\mathcal{H}_i'$ for $i = 1, 2, 3$ are defined as $\mathbf{H}_i$ replacing the term super-martingale by martingale.

Schachermayer (2002) proves the following theorem:

Theorem 3.2.13. *Let $S = (S_t)_{t\in[0,T]}$ be a locally bounded $\mathbb{R}^d-$valued semi-martingale and $\mathcal{M}_e \neq \emptyset$. U a utility function of type II and $\hat{Q}(y)$ the unique minimizer of the dual problem 3.86. For $x \in \mathbb{R}$, let*

$$u_i(x) = \sup_{H\in\mathbf{H}_i} E(U(x + \int_0^T N_u dS_u)).$$

Then the optimal solution $\hat{N}_i \in \mathbf{H}_i$ uniquely exists, i.e. the process $(\int_0^t N_u dS_u)_t$ is unique up to indistinguishability. The solution coincides in all cases $\hat{N} := \hat{N}_i$. Further, $\hat{N}$ is a unique optimizer in class $\mathbf{H}_1'$. Furthermore:

$$\forall i : u_i(x) = \sup\{E(U(x + \int_0^T N_u dS_u)),\ \text{N admissable}\} \tag{3.92}$$

Letting $y = u'(x)$ the following duality relation between $\hat{N}$ and $\hat{Q}(y)$ holds:

$$x + \int_0^T \hat{N}_u dS_u = I(y\frac{d\hat{Q}}{dP}),\ y\frac{d\hat{Q}(y)}{dP} = U'(x + \int_0^T \hat{N}_u dS_u) \tag{3.93}$$

In the case of the exponential utility function $U(x) = -e^{-\alpha x}$, $\hat{N}$ does not depend on x; it is also the unique minimizer in the classes $\mathbf{H}_2'$ and $\mathbf{H}_3'$ and as in (3.69) and (3.70), (3.93) become:

$$x + \int_0^T \hat{N}_u dS_u = -\frac{1}{\alpha}\log(\frac{y}{\alpha}\frac{d\hat{Q}}{dP}),\ \frac{d\hat{Q}(y)}{dP} = \frac{\alpha}{y}e^{-\alpha(x+\int_0^T \hat{N}_u dS_u)} \tag{3.94}$$

Again the maximum has the same form as in Theorem 3.2.12 and corresponds to our previous considerations.

3.2.3 Some Comments on the Hedging Problem

In this section, we give different ideas how to solve the hedging problem (problem (2.3)), when the problem is already solved for $\xi \equiv 0$. The section does not include an exact treatment of the problem. The focus lies on the ideas.

Recall, the hedging problem is of the form:

$$\sup_{Y\in\mathcal{W}_i} E_P(U(Y_T - \xi)) \tag{3.95}$$

on an arbitrary probability space $(\Omega, \mathcal{F}, P)$. The first device to solve (3.95) is to find an equivalent measure P_ξ such that problem (3.95) is equivalent to:

$$\sup_{Y \in \mathcal{W}_i} E_{P_\xi}(U(Y_T))$$

We have to separate Y_T and ξ. We look for a function h such that $U(Y_T - \xi) = U(Y_T)h(\xi)$ and set:

$$dP_\xi = (E(h(\xi)))^{-1} h(\xi) dP$$

This is possible, if U is the exponential utility function, as seen in section 3.2.2. However, we have to take care that the set we optimize over in the corresponding dual problem are equal. Then we can use the approach for $\xi \equiv 0$. Usually we do not have such a nice separation property. However, for example in the case of iso-elastic utility functions with parameter $p > 1$, one can separate $U(Y_T - \xi)$ such that:

$$U(Y_T - \xi) = \tilde{U}(Y_T)\tilde{h}(\xi),$$

where $\tilde{U}(Y_T) = \sum a_j Y_T^j$. Then, we can try to solve the problem $E(\tilde{U}(Y_T))$.

The second approach relies on the observation that (3.95) is equivalent to:

$$\sup_{(N,C) \in \mathcal{A} \times \mathcal{K}} E_P(U(\int_0^T N dS - C_T - \xi)) \tag{3.96}$$

$\mathcal{A} \times \mathcal{K}$ is isomorph to $\tilde{\mathcal{A}} \times \mathcal{K}$, where

$$\tilde{\mathcal{A}} = \mathcal{A} \times \{-1\}.$$

We set $S^V = (S, V)$ with $V_T = \xi$ (e.g. $V_t = E(\xi | \mathcal{F}_t)$), then (3.96) is equal to:

$$\sup_{(N,C) \in \tilde{\mathcal{A}} \times \mathcal{K}} E_P(U(\int_0^T N dS^V - C_T)) \tag{3.97}$$

Unfortunately, the dual problem is not the same in general. However, in the cases when

$$\mathcal{M}_e(S) = \mathcal{M}_e(S^V) \tag{3.98}$$

we can treat (3.97) in the same way as (3.95) for $\xi \equiv 0$. Recall, $\mathcal{M}_e(S)$ denotes the equivalent martingale measures with respect to S. So if we can prove (3.98), we can reduce (3.95) to the case when $\xi \equiv 0$. We give an idea and touch utility indifference pricing. Suppose, (3.95) has a solution for the class of wealth processes $\mathcal{W}$ as well as for the class of wealth processes generated by S^V, denoted by $\mathcal{W}^{S^V}$ and the class of the terminal values of $\mathcal{W}^{S^V}$. The solution of (3.95) with respect to $\mathcal{W}$ ($\mathcal{W}^{S^V}$) is called X_0^S ($X_0^{S^V}$). Further, denote Z_T^{opt} the optimal solution of the dual problem with respect to $\mathcal{W}$ and $Z_t^{opt} = E(Z_T^{opt} | \mathcal{F}_t)$. If now,

$$EU(X_0^S) \neq EU(X_0^{S^V}) \tag{3.99}$$

then $\mathcal{M}_e(S) \neq \mathcal{M}_e(S^V)$ in general since $Z_{opt} \notin \mathcal{M}_e(S^V)$: This is a consequence of Proposition 1.5.17. in Leitner (2001) for special utility functions $(\mathcal{C}^p)$ including isoelastic utility functions with parameter $p > 1$:

Proposition 3.2.14. *Let $V \in \mathbb{A}$, $\xi \in L^p(P)$ and $V_T = \xi$. Then (3.99) holds if and only if $VZ_{opt} \in \mathcal{U}$. In particular, $V_t = \frac{E(\xi Z_{opt}|\mathcal{F}_t)}{Z_t^{opt}}$ a.s. on $\{Z_t^{opt} \neq 0\}$ for all $t \geq 0$.*

The proof only uses that a solution to problem 3.95 with $\xi \equiv 0$ exists. Hence the result can be easily generalized to any strictly concave utility function, provided the solution to problem 3.95 with $\xi \equiv 0$ exists. So if 3.98 does not hold, then $\mathcal{M}_e(S) \neq \mathcal{M}_e(S^V)$. If 3.98 holds, we have derived a price. This approach is called utility indifference pricing, see also Davis (1997).

Thirdly, suppose we are in a market where a martingale representation result holds, i.e. for every martingale N there exists a predictable process H such that $N_t = N_0 + \int HdM$, where M is the martingale part of the stock S. We set:

$$P_t := Y_t - E[\xi|F_t] = Y_t - \int HdM$$

provided the conditional expectation of ξ exists. This changes the dynamics of Y to P. It has to be ensured that P keeps the same form as Y. This approach is used in Kohlmann and Zhou (2000) in the Brownian case using a quadratic utility function. The optimal control is explicitly derived using that every martingale in a Brownian setting can be represented by $\sum_{j=1}^{d} Z_j(t)dW^j(t)$. However, they start with the controlled system of the form:

$$dY(t) = (r(t)Y(t) + \sum_{j=1}^{d} \mu_j(t)u_j(t) + f(t))dW^j(t) + \sum_{j=1}^{d} u_j(t)dW^j(t), Y(0) = x$$

In correspondence to our Brownian model, $u' = \sigma'\pi$. Since σ' is not invertible it is not clear that we can derive an optimal portfolio π from the obtained optimal control u.

Chapter 4

Different Possible Solutions of the Dual Problem

In section 2.2.2, we introduced: $\mathcal{E}(-\int_0^T \bar{\theta}_s dW_s)$, where $\bar{\theta} = \sigma'(\sigma\sigma')^{-1}\mu$. It turns out to be the minimal martingale measure in the Brownian model, as we will see in the first section. In a complete market, we have a unique martingale measure to price a claim. However, which martingale measure should we take in the incomplete case? A perfect hedge for every contingent claim is not possible in such a market. A hedging argument as in the complete case (perfect hedge) is not sufficient to find a price in an incomplete market. So one possibility to find a price is to choose a pricing measure Q_{max} according to a certain criterion and take $E^{Q_{max}}(\xi)$ as the price in a discounted market. The second approach is a utility maximization approach. We define a utility function which represents the preference structure and maximizes $E(U(Y_T - \xi))$, where U is still a concave function (equivalently we could use a convex cost functional and minimize). The initial wealth necessary to finance the hedging or superhedging strategy corresponding to the minimum is then another model for the price. We have seen in the previous chapters that utility functions correspond to certain measures via the dual problem, e.g. the exponential utility function to the minimal entropy measure. If we fix a contingent claim and suppose we obtain the same dual solution for different utility functions then the price equals the expectation of the contingent claim under this dual solution and is therefore equal for every utility function (see e.g. Leitner (2001) mean-variance-hedging). However, the measure can change if the contingent claim changes. This yields different pricing measures for different contingent claims. Dual solutions are different.

So in the first section, we focus on several properties of different pricing measures or dual solutions, respectively. Their connection to the primal problem is not considered. We start with the minimal martingale measure in different settings. We further continue with the minimal entropy and

q-optimal measures and their interconnections. Section 4.2 discusses convergence results of q-optimal measures to the minimal entropy measure when q is tending to 1.

4.1 Optimal Pricing Measures

4.1.1 Minimal Martingale Measure

One approach to choose a martingale measure is taking a measure which preserves the structure of P best. This observation leads to the definition of the minimal martingale measure, which we characterize in the sequel. For further details see Föllmer and Schweizer (1991) or Schweizer (1995). We first define the minimal martingale measure for one stock in a general semimartingale model via a minimal property within the set (denoted by $\mathcal{M}^2_{e,R}$) of all equivalent martingale measures Q with a square-integrable density Z_T and S a true $Q-$martingale (Definition 4.1.2). We show that the martingale density is in this case equivalent to the measure generated by $\hat{Z}_T$, see (2.18). The measure (2.18) is defined explicitly and not via properties of a measure within a certain class of measures. It is therefore independent of the set of martingale measures and an extension of the definition in Föllmer and Schweizer (1991) (Definition 4.1.2), since the density of the measure given in definition 4.1.2 has exactly the form of $\hat{Z}_T$ given in (2.18). We explain the concept in a Brownian model. We show that the concept in definition 4.1.2 can be reduced to local not necessarily square-integrable martingales. Afterwards we extend the definition to the case of n stocks and again apply the result to the Brownian case. The most general definition including all others is already given (2.18). We call this measure the minimal martingale measure and use it in the following chapters.

Minimal Martingale Measure - One-Dimensional Case

We start with the general semimartingale model for a stock S:

$$S_t = S_0 + M_t + A_t, \tag{4.1}$$

where M is continuous. We define:

Definition 4.1.1. The set of all equivalent measures Q with density Z_T and S a real $Q-$martingale is denoted by $\mathcal{M}_{e,R}$. If the density is in addition square-integrable, we denote the space by $\mathcal{M}^2_{e,R}$.

So we temporarily define the minimal martingale measure in the class $\mathcal{M}^2_{e,R}$:

Definition 4.1.2. An equivalent martingale measure $Q_{mmm} \in \mathcal{M}^2_{e,R}$ is called minimal in the class $\mathcal{M}^2_{e,R}$ (Minimal Martingale Measure in $\mathcal{M}^2_{e,R}$), if

1. $Q_{mmm} = P$ on $\mathcal{F}_0$
2. Any square-integrable P-martingale which is orthogonal to the P-martingale part M of the stock S remains a martingale under Q_{mmm}, i.e.

$$N \in \mathbb{M}^2, \langle N, M \rangle = 0 \Rightarrow N \text{ is a martingale under } Q_{mmm}$$

In the one dimensional case, the minimal martingale measure is characterized in Föllmer and Schweizer (1991), Theorem 1:

Theorem 4.1.1. *The minimal martingale measure Q_{mmm} is uniquely determined. It exists in $\mathcal{M}^2_{e,R}$ if and only if*

$$\hat{Z}_t = \exp\{-\int_0^t \alpha'_s dM_s - \frac{1}{2}\int_0^t \|\alpha_s\|^2 d\langle M\rangle_s\},$$

is a square-integrable martingale under P, where $A_t = \int^t \alpha'_s d\langle M\rangle_s$; in this case, it holds: $\frac{dQ_{mmm}}{dP} = Z^{mmm}_T = \hat{Z}_T$. Finally, the minimal martingale measure preserves orthogonality: For all $L \in \mathbb{M}^2$ satisfying $\langle L, M\rangle = 0$ under P also $\langle L, S\rangle = 0$ holds under Q_{mmm}.

So $\hat{Z}_T$ defined in (2.18) is an extension of definition 4.1.2.

We illustrate this definition in the Brownian Model:

Brownian case (one stock)

We apply the previous results to:

$$S_t = S_0 + \int_0^t \mu(s, S_s)ds + \int_0^t \sigma(s, S_s)dW_s, \tag{4.2}$$

where W is a $d-$dimensional Brownian Motion and σ a $1 \times d-$ matrix, so that we end up with a model for a single stock and one bond.

Note first, in the general semimartingale model: $S_t = S_0 + M_t + A_t$, we required that $Q_{mmm} = P$ on $\mathcal{F}_0$. This is trivially satisfied in the Brownian case, since $(\mathcal{F}_t)_t$ is the augmentation of the filtration generated by the chosen d-dimensional Brownian motion W. $\mathcal{F}_0$ is generated by the random variable $W_0 = 0$ and hence $\mathcal{F}_0 = \{\Omega, \emptyset, \mathrm{N}\}$. N denote the set of sets with measure zero. There are needed to satisfy the assumption of completeness. The set is equal under both measures, because both measures are equivalent.

In the context of a multidimensional Brownian Model, the minimal martingale measure has to be defined as follows:

Definition 4.1.3. A P-martingale measure $Q_{mmm} \in \mathcal{M}^2_{e,R}$ is called minimal if any square-integrable P-martingale, that is orthogonal to $\int_0^t \sigma(s, S_s)dW_s, t \in [0, T]$ under P, is a martingale under Q_{mmm}.

Theorem 4.1.1 deduces to:

Corollary 4.1.2. *The minimal martingale measure Q_{mmm} in definition 4.1.3 is uniquely determined. It exists if and only if*

$$\hat{Z}_t = \exp\{-\int_0^t \bar{\theta}'_s dW_s - \frac{1}{2}\int_0^t \|\bar{\theta}_s\|^2 ds\},$$

where $\bar{\theta} = \sigma'(\sigma\sigma')^{-1}\mu$, is a square-integrable martingale under P; in this case, it holds: $\frac{dQ_{mmm}}{dP} =: Z_T^{mmm} = \hat{Z}_T$. $\alpha'_s = \mu'(\sigma\sigma')^{-1}$. Finally, the minimal martingale measure preserves orthogonality: For all $L \in \mathbb{M}^2$ satisfying $\langle L, \int \sigma dW\rangle = 0$ under P also $\langle L, S\rangle = 0$ holds under Q_{mmm}.

Proof. We only have to prove the adjustment to Theorem 4.1.1.

$$A_t = \int_0^t \alpha'_s \sigma_s \sigma'_s ds = \int_0^t \mu'_s I ds,$$

by a comparison of the coefficients: $\alpha' = \mu'(\sigma\sigma')^{-1}$. Further, we see:

$$\int_0^t \alpha'_s dM_s = \int_0^t \alpha'_s \sigma_s dW_s = \int_0^t \mu'_s(\sigma_s\sigma'_s)^{-1}\sigma_s ds = \int_0^t \bar{\theta}'_s dW_s$$

□

Note, α and μ are one-dimensional to prepare for the multidimensional case (finite number n of stocks), we take care of the transposed signs. The measure generated by $\hat{Z}_T = \mathcal{E}_T(-\int \bar{\theta}' dW)$ (see 2.18) is not only an extension of definition 4.1.3, it also generalizes the following definition extending definition (see also El Karoui and Quenez (1995)):

Definition 4.1.4. A P-martingale measure $Q_{mmm} \in \mathcal{M}_{e,R}$ is called minimal if any local P-martingale, that is orthogonal to $\int_0^t \sigma(s, S_s)dW_s, t \in [0,T]$ under P, is a local martingale under Q_{mmm}.

It is an extension since:

Lemma 4.1.3. *The parameter $\bar{\theta}$ generates the minimal martingale measure in definition 4.1.4, i.e. $Z_{mmm} = \mathcal{E}_T(-\int \bar{\theta} dW_s)$, if and only if*

$$\bar{\theta} \in im\, \sigma'_t \Leftrightarrow \bar{\theta} = \sigma'(\sigma\sigma')^{-1}\mu$$

We prove the lemma later for n-stocks.

Next, we consider a special case of $\sigma(t, S_t) = S_t \cdot \sigma^c(t)$, where σ^c is deterministic:

$$S_t = S_0 + \int_0^t S_s \mu_s ds + \int_0^t S_s \sigma^c_s dW_S = S_0 + A_t + M_t \qquad (4.3)$$

Here, we have the problem that $\int_0^t S_s\sigma_s^c dW_s$ might explode and therefore might not be square-integrable. That turns out not to be a problem. Suppose it is square-integrable, then we obtain the same measure when considering $\tilde{S} = \log S = S_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s^c dW_S$. So we can just switch to $\tilde{S}$. As before (set $\sigma(t,\tilde{S}) = \sigma^c$), the minimal martingale measure of $\tilde{S}$ is of the form:

$$Z_{mmm}^{\log S} = \exp\{-\int_0^t \bar{\theta}_s' dW_s - \frac{1}{2}\int_0^t \|\bar{\theta}_s\|^2 ds\},$$

where $\bar{\theta} = (\sigma^c)'((\sigma^c)(\sigma^c)')^{-1}\mu$. On the other hand for S we have:

$$A_t = \int_0^t \alpha_s'\sigma(s,S_s)\sigma(s,S_s)'ds = \int_0^t \alpha_s' S_s \cdot \sigma_s^c(\sigma_s^c)' \cdot S_s' ds = \int_0^t \mu_s' S_s ds,$$

by a comparison of the coefficients: $\alpha' = \mu'(\sigma\sigma')^{-1}S^{-1}$. Further, we see:

$$\int_0^t \alpha_s' dM_s = \int_0^t \alpha_s \sigma_s^c S_s dW_s = \int_0^t \mu_s'(\sigma_s^c(\sigma_s^c)')^{-1}\sigma_s^c S_s^{-1} S_s ds = \int_0^T \bar{\theta}_s' dW_s,$$

where $\bar{\theta} = (\sigma^c)'((\sigma^c)(\sigma^c)')^{-1}\mu$. Hence by Theorem 4.1.1,

$$Z_{mmm}^{S} = \exp\{-\int_0^t \bar{\theta}_s' dW_s - \frac{1}{2}\int_0^t \|\bar{\theta}_s\|^2 ds\},$$

So the minimal martingale measure of $\log S$ is equal to the minimal martingale measure generated by S and is therefore an extension of the minimal martingale measure of S.

Minimal Martingale Measure - Multidimensional Case

We now extend the minimal martingale measure to the case of n stocks. Using a multidimensional version of Lemma 4.1.3, we can define the minimal martingale measure more generally. We follow the approach of Schweizer (1995), using Theorem 2.2.6 and the corresponding discussion:

Definition 4.1.5. The measure $\hat{Q}$ defined by

$$\frac{d\hat{Q}}{dP} = \hat{Z}_T, \quad \hat{Z}_T = \mathcal{E}_T(-\int \hat{\lambda}' dM)$$

is called the minimal martingale measure, where $\hat{\lambda}$ is defined in (2.17).

Before, we show that the structure condition is satisfied in the Brownian setting and that $\hat{Z}$ corresponds to $\mathcal{E}(\int \bar{\theta}_s' dW_s)$, we give a definition of the minimal martingale measure extending definition 4.1.2 to the case of n stocks (definition 4.1.6). Theorem 1 in Föllmer and Schweizer (1991) can be easily extended to the multidimensional case. So the measure defined in definition 4.1.6 is equivalent to definition 4.1.5, if it is square-integrable (Theorem 1 in the case of one stock). In the following chapters, we denote minimal martingale measure by Z_{mmm} and meaning the extension $\hat{Z}_T$, if it does not exist in the original sense (Definition 4.1.2 and 4.1.6). Note, Föllmer and Schweizer (1991) work with continuous processes. So we restrict ourself to this case. We give a second definition of the minimal martingale measure:

Definition 4.1.6. An equivalent martingale measure $Q_{mmm} \in \mathcal{M}^2_{e,R}$ is called minimal (Minimal Martingale measure), if

1. $Q_{mmm} = P$ on $\mathcal{F}_0$
2. Any square-integrable P-martingale N which is orthogonal to each component of the P-martingale part of the stock M_t remains a martingale under Q_{mmm}, i.e.

$$N \in \mathbb{M}^2, \forall i \leq n: \ \langle N, M^{(i)} \rangle = 0 \Rightarrow N \text{ is a martingale under } Q_{mmm}.$$

Next, we extend Theorem 4.1.1, which shows the equivalence of definition 4.1.5 and 4.1.6 in the case that $\hat{Z}_T$ is square-integrable. The measure generated by $\hat{Z}_T$ is therefore an extension of the minimal martingale measure defined in definition 4.1.6:

Theorem 4.1.4. *The minimal martingale measure Q_{mmm} defined in definition 4.1.6 is uniquely determined. It exists in $\mathcal{M}^2_{e,R}$ if and only if*

$$\hat{Z}_t = \mathcal{E}_t(-\int \hat{\lambda}' dM) = \exp\{-\int_0^t \hat{\lambda}'_s dM_s - \frac{1}{2}\int_0^t \|\hat{\lambda}\|^2 d\langle M\rangle_s\},$$

where $\hat{\lambda}$ is implicitly defined as in (2.17), is a square-integrable martingale under P; in this case, it holds: $\frac{dQ_{mmm}}{dP} =: Z_T^{mmm} = \hat{Z}_t$. Finally, the minimal martingale measure preserves orthogonality: For all $L \in \mathbb{M}^2$ satisfying $\forall i$: $\langle L, M^{(i)}\rangle = 0$ under P also holds $\langle L, S^{(i)}\rangle = 0$ under Q_{mmm}.

Proof. The proof of the last assertion is analogously to the proof in Föllmer and Schweizer (1991), Theorem 1. They show that $\langle N, S\rangle = 0$ under Q_{mmm} if $\langle N, M\rangle = 0$ under P for a square-integrable martingale. S and M are one-dimensional, so we just have to set $M^{(i)} = M$ and $S^{(i)} = S$ in our case and follow the exact same procedure and use that the martingale part M of S is continuous.

We only show the remaining claims. Firstly, we show general properties of an arbitrary $\tilde{Z}_T \in \mathcal{M}_Z^{e,R}$ and secondly we establish that if a minimal $\tilde{Z}_T$ exists then it is unique and $\tilde{Z}_T = \hat{Z}_T$, hence $\hat{Z}_T = Z_{mmm}$. Finally, we establish the above equivalence.

We use the fact that every considered density is assumed to be square-integrable, so $\tilde{Z}_t = E(\tilde{Z}_T|\mathcal{F}_t)$ is in $\mathcal{H}^2(P)$ by the inequalities of Doob and

Burkholder-Davis-Gundy. Thus, (see Föllmer and Schweizer (1991) G-K-W-decomposition):

$$\tilde{Z}_t = \tilde{Z}_0 + \int_0^t \beta_s' dM_s + L_t,\ t \in [0,T] \tag{4.4}$$

where L is orthogonal to every $M^{(i)},\ i \leq n$ under P and $(\beta_s)_{s\leq T}$ a predictable process with

$$E(\int_0^T \beta_s' d\langle M\rangle \beta_s) < \infty.$$

Under $\tilde{P}$ defined by $d\tilde{P} = \tilde{Z}_T dP$, the Doob-decomposition of M is of the form $M = \tilde{M} + \tilde{A}$, where $\tilde{A}$ is the part of bounded variation:

$$\tilde{A}_t = \int_0^t \tilde{Z}_{s^-}^{-1} d\langle Z, M\rangle_s = \int_0^t Z_{s^-}^{-1} \beta_s' d\langle M\rangle_s$$

So,

$$\hat{\lambda}_s' = -Z_{s^-}^{-1}\beta_s'$$

since $\tilde{Z}_T > 0,\ P - a.s.$ we have that

$$\int_0^T \hat{\lambda}_s' d\langle X\rangle_s \hat{\lambda}_s < \infty \ \ P - a.s$$

We now prove uniqueness. Suppose there exists a $\tilde{Z}$, which is minimal. We have that $\tilde{Z}_0 = 1$ and by the minimality (definition 4.1.6) L is a martingale under $\tilde{P}$. Hence, the part of bounded variation A_t^L of the Doob-Meyer-decomposition is zero:

$$\forall t:\ A_t^L = -\int_0^t Z_s^{-1} d\langle L, Z\rangle_s = 0,$$

thus $\forall t: \langle L, Z\rangle_t = 0$ and

$$\langle L, Z\rangle_t = \langle L, Z_0 + \int_0^{\cdot} \beta_s' dM_s + L\rangle = \langle L\rangle,$$

because of the orthogonality. From $\langle L\rangle = 0$ follows that $L = 0$ and therefore the uniqueness.

We now come to the equivalence:

"⇒:" We assume that Q_{mmm} exists. Since we have seen that $L = 0$, the only density which satisfying this is $\hat{Z}_T$. The square-integrability follows from the fact that $Q_{mmm} \in \mathcal{M}_{e,R}^2$.

"⇐:" Next we suppose that $\hat{Z}_t = \mathcal{E}_t(-\int \hat{\lambda}' dM)$ is square-integrable under P and show that it is in $\mathcal{M}_{e,R}^2$ and minimal. We consider an $N \in \mathcal{M}^2$ with $\langle N, M\rangle = 0$ under P. Since $\hat{Z} = Z_0 + \int^{\cdot} (\hat{Z})^{-1} dM$, we have that

$\langle N, \hat{Z} \rangle = 0$, which yields that the part of bounded variation under Q_{mmm} is zero:

$$A_t = \int^{\cdot} (\hat{Z})^{-1} d\langle N, \hat{Z}_t \rangle = 0,$$

it follows that N is a local martingale under Q_{mmm}, since by Itô's formula:

$$dL\hat{Z} = Ld\hat{Z} + \hat{Z}dL + d\langle L, \hat{Z} \rangle.$$

N is in $H^2(P)$, hence

$$\sup_{t \le T} |N_t|^2 \in L^1(P) \Rightarrow \sup_{t \le T} |N_t| \in L^2(P) \Rightarrow \sup_{t \le T} |N_t| \in L^1(Q_{mmm})$$

by Hölder's inequality using that $\hat{Z}_T \in L^2(P)$. Hence, $L \in \mathcal{H}^1(Q)$ and the local Q-martingale L is therefore a Q-martingale. So the minimal martingale measure uniquely exists if and only if $\hat{Z}_T$ is square-integrable under P. $\square$

Finally, $\hat{Z}_T = \mathcal{E}(-\int_0^T \hat{\lambda}' dM)$ in definition 4.1.5 and Z_{mmm} defined in definition 4.1.2 and 4.1.6, respectively are the same in the case that $\hat{Z}_T$ is square-integrable. Otherwise the measure defined in definition 4.1.2 and 4.1.6 does not exists in the class $\mathcal{M}^2_{e,R}$. $\hat{Z}_T$ is therefore an extension of Z_{mmm}.

Brownian case (n stocks)

We now come back to our Brownian setting. We consider an n-dimensional stock:

$$S_t = S_0 + \int_0^t \mu(s, S_s) ds + \int_0^t \sigma(s, S_s) dW_s, \tag{4.5}$$

where W is a $d-$dimensional Brownian Motion and σ a $n \times d-$ matrix, so that we end up with a model for n stocks.

We show that $\hat{Z}_T = \mathcal{E}(-\int_0^T \bar{\theta}' dW_s)$, which corresponds to definition 4.1.2 in the case of one stock, in general is an extension of the following definition as a special case of definition 4.1.6:

Definition 4.1.7. A P-martingale measure $Q_{mmm} \in \mathcal{M}^2_{e,R}$ is called minimal if any square-integrable P-martingale that is orthogonal to $\int_0^t \sigma_s^{(i)} dW_s, t \in [0, T], i \le n$ under P is a martingale under Q_{mmm}.

Again Theorem 4.1.4 holds in the Brownian case:

Corollary 4.1.5. *The minimal martingale measure Q_{mmm} of definition 4.1.7 is uniquely determined. It exists if and only if*

$$\hat{Z}_t = \exp\{-\int_0^t \bar{\theta}_s' dW_s - \frac{1}{2}\int_0^t \|\bar{\theta}_s\|^2 ds\},$$

where $\bar{\theta} = \sigma'(\sigma\sigma')^{-1}\mu$, *is a square-integrable martingale under* P*; in this case, it holds:* $\frac{dQ_{mmm}}{dP} =: Z_T^{mmm} = \hat{Z}_T$, *where* $\hat{\lambda}'_s = \mu'(\sigma\sigma')^{-1}$. *Finally, the minimal martingale measure preserves orthogonality: For all* $L \in \mathcal{M}^2$ *satisfying* $\forall i: \langle L, \int \sigma^{(i)} dW\rangle = 0$ *under* P *also holds* $\langle L, S^{(i)}\rangle = 0$ *under* Q_{mmm}.

To establish the equivalence of definition 4.1.5 and 4.1.7 (corollary 4.1.5), by Theorem 4.1.4 it remains to show that:

$$\hat{Z} = \mathcal{E}(\int \hat{\lambda}'_s dM) = \mathcal{E}(-\int \bar{\theta}' dW_s) \tag{4.6}$$

We calculate $\hat{Z} = \mathcal{E}(-\int \hat{\lambda}' dM)$ to show 4.6 and start with $d\langle M\rangle$:

Lemma 4.1.6. *Suppose an n-dimensional stock is given by (4.7), then we have:*

$$d\langle M\rangle = \sigma\sigma' dt$$

Proof. We have

$$\begin{aligned} d\langle M\rangle &:= d\langle M, M\rangle = \langle \sigma dW, \sigma dW\rangle \\ &= (\sigma dW)(\sigma dW)' \\ &= \sigma(dW dW')\sigma' \\ &= \sigma I_d \sigma' dt \end{aligned}$$

More precisely:

$$\begin{aligned} d\langle M^{(i)}, M^{(k)}\rangle_t &= \langle \sum_j \sigma_{ij} dW^j, \sum_l \sigma_{kl} dW^l\rangle \\ &= \sum_j \sum_l \sigma_{ij}\sigma_{kl}\langle dW^j, dW^l\rangle \\ &= \sum_j \sigma_{ij}\sigma_{kj} dt \end{aligned}$$

the last inequality follows since $\langle W^j, W^k\rangle = 0, j \neq k$ and $\langle W^j, W^k\rangle = t, j = k$. □

So $\hat{\lambda}$ exists and is unique since $\sigma\sigma'$ is positive definite. So we showed directly that the structural condition is satisfied. Next, we determine $\hat{\lambda}$, we have

$$A_t^{(i)} = \int^t \mu_s^{(i)} ds = \int^t \alpha_s^{(i)} d\langle M^{(i)}\rangle_s$$

According to the definition of $\hat{\lambda}$, we have:

$$A_t = \int^t \mu'_s \mathbf{I} ds = \int^t \hat{\lambda}'_s d\langle M\rangle.$$

So it must hold:

$$\mu_s'\mathbf{I} = \hat{\lambda}_s'\mathbf{I}\sigma_s\sigma_s'\mathbf{I}$$

We generally assume that a stock price is strictly positive:

$$\mu_s'(\sigma_s\sigma_s')^{-1}\mathbf{I} = \hat{\lambda}_s'$$

And therefore

$$\begin{aligned}\int_0^T \hat{\lambda}_s' dM &= \int_0^T \mu_s'(\sigma_s\sigma_s')^{-1}\mathbf{I}^{-1}\mathbf{I}\sigma_s dW_s \\ &= \int_0^T \bar{\theta}_s' dW_s.\end{aligned}$$

Hence,

$$\mathcal{E}(\int_0^T \hat{\lambda}_s' dM) = \mathcal{E}(\int_0^T \bar{\theta}_s' dW_s)$$

If the coefficients σ and μ are bounded then $\bar{\theta}$ is bounded, hence the corresponding density is square-integrable with respect to all martingale measures and P. So the minimal martingale measure in Schweizer (1995) (definition 4.1.5) coincides with Definition 4.1.7 in the Brownian case. Together with Theorem 4.1.4, this proves Corollary 4.1.5.

As in the case of one stock, we can define:

Definition 4.1.8. A P-martingale measure $Q_{mmm} \in \mathcal{M}_{e,R}$ is called minimal if any local P-martingale that is orthogonal to $\int_0^t \sigma_s^{(i)} dW_s, t \in [0,T], i \leq n$ under P is a local martingale under Q_{mmm}.

This can be justified by the following extension of Lemma 4.1.3:

Lemma 4.1.7. *The parameter $\bar{\theta}$ generates the minimal martingale measure of definition 4.1.8, i.e. $Z_{mmm} = \mathcal{E}_T(-\int \bar{\theta}' dW_s)$, if and only if*

$$\bar{\theta} \in im\, \sigma_t' \Leftrightarrow \bar{\theta} = \sigma'(\sigma\sigma')^{-1}\mu$$

We sketch the proof of Lemma 4.1.7, see also El Karoui and Quenez (1995), Proposition 1.8.2.:

Proof. For the last equivalence see Kohlmann (2003) Lemma 2.5.52. $\theta_s \in im(\sigma')$ is uniquely determined by $\sigma'(\sigma\sigma')^{-1}\mu$. The first equivalence is clear by theorem 4.1.1 in the one-dimensional case. However, we show theorem 4.1.1 again for n stocks in case of a Brownian model. We start with a P-square-integrable martingale N. By the martingale representation theorem, we have $N_t = N_0 + \int^t f_s dW_s$ for a square-integrable f, further denote $M_i =$

$\int^t \sigma_s^{(i)} dW_s,\ 1 \leq i \leq n$ where $\sigma^{(i)}$ the i^{th} row of σ. Furthermore, by Girsanov's theorem we have:

$$N_t = N_0 + \int_0^t f_s dW_s^\theta - \int_0^t f_s \theta_s ds$$

Since f is square-integrable, N remains a martingale under Q^θ if and only of θ_s is orthogonal to f. If θ_s generates the minimal martingale measure, we have that

$$\begin{aligned} &\forall t\ \forall i \leq n:\ \langle N, M_i \rangle = \int \sigma_s^{(i)} f_s ds = 0 \\ \Leftrightarrow\quad &\sigma_s f_s = 0 \Leftrightarrow f \in ker(\sigma) \end{aligned}$$

Since θ is orthogonal to f:

$$\forall t\ \forall i \leq n:\ \langle N, M_i \rangle = 0 \Leftrightarrow \theta_s \in im(\sigma')$$

□

We again extend the special case from before:

$$S_t = S_0 + \int_0^t \mu_s' \mathbf{S}_s ds + \int_0^t \mathbf{S}_s \sigma_s^c dW_s = S_0 + M_t + A_t \tag{4.7}$$

where $\mu = (\mu^{(1)}, ..., \mu^{(n)})'$ and

$$\mathbf{S}_{n \times n} = \begin{bmatrix} S^{(1)} & & 0 \\ & ... & \\ 0 & & S^{(n)} \end{bmatrix}$$

where $S^{(i)}$ denotes the i^{th} stock.

Remark 4.1.1. (Theorem 4.1.7) The proof of Theorem 4.1.7 almost stays the same if S does not explode. We just replace M_i by $\int^t S_s^{(i)} (\sigma_s^c)^{(i)} dW_s$, then $S_s^{(i)}$ does not play a role in $\langle N, M_i \rangle = \int (\sigma_s^c)^{(i)} f_s S_s^{(i)} ds = 0$.

□

We again calculate: $\hat{Z} = \mathcal{E}(-\int \hat{\lambda}' dM)$ and find out that both measures again coincide ($\log S$ and S.)

Analogously, for $d\langle M \rangle$, we get:

Lemma 4.1.8. *Suppose an n-dimensional stock is given by (4.7), then we have:*

$$d\langle M \rangle = \mathbf{S} \sigma^c (\sigma^c)' \mathbf{S} dt$$

Proof. Set $\sigma = \sigma^c$. We have

$$\begin{aligned} d\langle M\rangle : &= d\langle M, M\rangle = \langle \mathbf{S}\sigma dW, \mathbf{S}_s\sigma dW\rangle \\ &= (\mathbf{S}\sigma dW)(\mathbf{S}\sigma dW)' \\ &= (\mathbf{S}\sigma dW)(\sigma dW)'\mathbf{S} \\ &= \mathbf{S}\sigma(dW dW')\sigma'\mathbf{S} \\ &= \mathbf{S}\sigma I_d \sigma'\mathbf{S}dt \end{aligned}$$

More precisely:

$$\begin{aligned} d\langle M^{(i)}, M^{(k)}\rangle_t &= \langle \sum_j S^{(i)}\sigma_{ij}dW^j, \sum_l S^{(k)}\sigma_{kl}dW^l\rangle \\ &= \sum_j \sum_l S^{(i)}\sigma_{ij}S^{(k)}\sigma_{kl}\langle dW^j, dW^l\rangle \\ &= \sum_j S^{(i)}\sigma_{ij}S^{(k)}\sigma_{kj}dt \end{aligned}$$

the last inequality follows since $\langle W^j, W^k\rangle = 0, j \neq k$ and $\langle W^j, W^k\rangle = t, j = k$. □

So $\hat{\lambda}$ exists and is unique since $\sigma^c(\sigma^c)'$ is positive definite and $\mathbf{S}^{-1}$ anyway. So we showed directly that the structural condition is satisfied. Next, we determine $\hat{\lambda}$, we have

$$A_t^{(i)} = \int^t \mu_s^{(i)} S_s^{(i)} ds = \int^t \alpha_s^{(i)} d\langle M^{(i)}\rangle_s$$

According to the definition of $\hat{\lambda}$, we have:

$$A_t = \int^t \mu_s' \mathbf{S}_s ds = \int^t \hat{\lambda}_s' d\langle M\rangle.$$

So it must hold:

$$\mu_s'\mathbf{S}_s = \hat{\lambda}_s'\mathbf{S}\sigma_s^c(\sigma_s^c)'\mathbf{S}_s$$

We generally assume that a stock price is strictly positive:

$$\mu_s'(\sigma_s^c(\sigma_s^c)')^{-1}\mathbf{S}_s^{-1} = \hat{\lambda}_s'$$

And therefore

$$\begin{aligned} \int_0^T \hat{\lambda}_s' dM &= \int_0^T \mu_s'(\sigma_s^c(\sigma_s^c)')^{-1}\mathbf{S}_s^{-1}\mathbf{S}_s\sigma_s^c ds \\ &= \int_0^T \bar{\theta}_s' dW_s, \end{aligned}$$

where $\bar{\theta} = (\sigma^c)(\sigma^c(\sigma^c)')^{-1}\mu$. Hence,

$$\mathcal{E}(\int_0^T \hat{\lambda}_s' dM) \quad = \quad \mathcal{E}(\int_0^T \bar{\theta}_s' dW_s)$$

and

$$Z_{mmm}^S \quad = \quad \mathcal{E}(\int_0^T \bar{\theta}_s' dW_s) = Z_{mmm}^{\log S}$$

4.1.2 Minimal Entropy Measure

We now consider the minimal entropy measure and explain its connection to the minimal martingale measure, e.g. we discuss the question when they coincide. We define the mean-variance process - a function of the martingale part of the stock. Using this definition, we give a characterization of the minimal martingale measure, first established in Föllmer and Schweizer (1991). It will be the minimizer of the entropy minus a penalty term. Depending on the form of the mean-variance process, this penalty does not depend on the measure. We obtain conditions when the minimal entropy and the minimal martingale measure coincide.

From section 1.1, we know that the entropy $H(Q|P)$ is zero if and only if $P = Q$. So if we want to construct a martingale measure which is in some way closest to P, we might take entropy as a measure of distance. Note, entropy is not a metric! This leads to the minimal entropy as a criterion. The solution is called the minimal entropy measure. In section 3.2.2 (Exponential Utility Function), we have seen that we have to ask for certain assumptions: To get a useful definition of a minimal entropy measure, we required that there exists at least one absolutely continuous martingale measure Q with finite entropy $H(Q|P)$, where P is the true measure of the underlying probability space (Ω, F, P). We therefore optimize over the set of all these measures satisfying the following requirement:

$$P_f(P) = \{Q \in \mathcal{M}_a | H(Q|P) < \infty\} \tag{4.8}$$

We state the following theorem from Frittelli (2002) (Theorem 2.1, 2.2):

Theorem 4.1.9. *$P_f(P) \neq \emptyset$ then there exists a unique local martingale measure Q_{min} with density Z_{min} that minimizes $H(Q|P)$ over all $Q \in P_f(P)$. If even $P_f(P) \cap \mathcal{M}_e \neq \emptyset$, then $Q_{min} \in \mathcal{M}_e$, i.e. equivalent to P.*

Note, this holds in a general semimartingale model with a d-dimensional stock S with continuous paths. We therefore define:

Definition 4.1.9. The minimal entropy measure is defined as the unique solution of:

$$\min_{Q \in P_f(P)} H(Q|P)$$

Recall, we always require that the n-dimensional stock S is continuous, consequently the structure condition is satisfied (see above or Schweizer (1995)). So there exists an $\mathbb{R}^n$-valued, continuous local P-martingale M and a predictable process $\hat{\lambda} \in L^2_{loc}(M)$ such that

$$S = S_0 + M + \int^{\cdot} d\langle M\rangle\hat{\lambda},$$

since:

$$A^{(i)} = \int \alpha^{(i)} d\langle M^{(i)}\rangle = \int \gamma^{(i)} = \int (d\langle M\rangle\hat{\lambda})^{(i)} \Rightarrow A = \int d\langle M\rangle\hat{\lambda}$$

Recall, $\hat{Z}_T = \mathcal{E}_T(\int \hat{\lambda}' dM)$ is called the minimal martingale measure. Further, from Theorem 2.2.6, we know that every equivalent martingale measure can be represented as follows:

$$\frac{dQ}{dP} = Z_Q,\ Z_Q = \mathcal{E}_T(M_Q), M^Q \in \mathbb{M}_{loc}$$

We introduce the notation:

$$\mathcal{E}_{tT}(M^Q) = \frac{\mathcal{E}_T(M^Q)}{\mathcal{E}_t(M^Q)},\ \langle M\rangle_{tT} = \langle M\rangle_T - \langle M\rangle_t \tag{4.9}$$

Mania et al (2003b) proved the following characterization of the minimal entropy measure (Theorem 3.1.):

Theorem 4.1.10. *Let all $(\mathcal{F}, P)$ local martingales be continuous and $P_{f,e}(P) := P_f(P) \cap \mathcal{M}_e \neq \emptyset$. Then the value process V_t given by*

$$V_t = \text{ess} \inf_{Q \in P_{f,e}(P)} E_Q(\log \mathcal{E}_{tT}(M^Q)|\mathcal{F}_t), \tag{4.10}$$

and is a special semimartingale with $V_t = m_t + A_t + V_0$, where $m \in \mathbb{M}^2_{loc}$, and A a predictable finite variation process. Therefore the G-K-W-decomposition exists:

$$m_t = \int_0^t \phi_s' dM_s + \tilde{m}_t,\ \langle \tilde{m}, M\rangle = 0. \tag{4.11}$$

Further, V_t is the first part of the solution of the following backward stochastic differential equation:

$$Y_t = Y_0 - \text{ess} \inf_{Q \in P_{f,e}(P)} (\frac{1}{2}\langle M^Q\rangle_t + \langle M^Q, L\rangle_t) + L_t,\ Y_T = 0 \tag{4.12}$$

Moreover, Q_{min} is the minimal entropy measure if and only if

$$\frac{dQ_{min}}{dP} = \mathcal{E}_T(M^{Q_{min}}),\ M_t^{Q_{min}} = -\int_0^t \hat{\lambda}_s' dM_s - \tilde{m}_t \tag{4.13}$$

Suppose, in addition, the minimal martingale measure exists, i.e. $\hat{Z}$ is a martingale, and satisfies the Log-Reverse-Hölder-inequality. Then, V uniquely solves the above BSDE (4.12) and is bounded.

Next, we give a characterization of the minimal martingale measure and define:

Definition 4.1.10. The process

$$\hat{K} = \langle -\int \hat{\lambda}' dM \rangle = \int \hat{\lambda}' d\langle M \rangle \hat{\lambda}$$

is called the mean-variance trade-off process.

For a detailed interpretation see Schweizer (1992) and Schweizer (1994). Föllmer and Schweizer (1991) first characterized the minimal martingale measure by the unique solution of:

$$\min_{Q \in P_f(P): E_Q(\hat{K}_T) < \infty} H(Q|P) - \frac{1}{2} E_Q(\hat{K}_T) \tag{4.14}$$

provided that $H(Q_{mmm}|P) < \infty$. $P_f(P)$ is a subset of all martingale measures and $H(Q|Q_{mmm}) = H(Q|P) - \frac{1}{2}E_Q(\hat{K}_T)$. Note, the solution of (4.14) is the result of minimizing the entropy minus a penalty term. $E_Q\hat{K}_T$ is positive, so if it is very low, the measure is penalized. Since the higher $E_Q(\hat{K}_T)$ the closer is $Z = \mathcal{E}(-\int \hat{\lambda}' dM)\mathcal{E}(N)$ to $\hat{Z}_T = \mathcal{E}(-\int \hat{\lambda}' dM)$. If $\hat{K}_T$ is deterministic the minimal martingale measure and the entropy measure are the same. Mania et al (2003b), (Corollary 3.4.) state conditions when the other direction is also true:

Theorem 4.1.11. *The mean variance process at T $\langle \int \hat{\lambda}' dM \rangle_T$ is deterministic if an only if the minimal entropy measure coincides with the minimal martingale measure and $\phi = 0$ $\mu^{\langle M \rangle} - a.e$, where ϕ is the integrand in the G-K-W decomposition with respect to M (see 4.27) and $\mu^{\langle M \rangle}$ the Doleans measure of $\langle M \rangle$.*

Theorem 4.1.11 is a Corollary of Proposition 3.2. in Mania et al (2003b). It gives a necessary and sufficient condition when the minimal entropy measure coincides with the minimal martingale measure.

Further suppose, that:

$$E(\exp(\frac{1+\epsilon}{2}\hat{K}_T)) < \infty \quad \text{for some} \quad \epsilon > 0 \tag{4.15}$$

and $\mathcal{M}_e \neq \emptyset$ and S is continuous, then by 4.14 we are able to characterize the minimal martingale measure by minimizing the entropy with respect to the measure $P_{\frac{1}{2\alpha}\hat{K}_T}$:

$$Q_{mmm} = \arg \min_{Q \in P_f(P)} H(Q|P_{\frac{1}{2\alpha}\hat{K}_T}), \tag{4.16}$$

where $P_{\frac{1}{2\alpha}\hat{K}_T}$ is defined by:

$$\frac{dP_{\frac{1}{2\alpha}\hat{K}_T}}{dP} = c \cdot e^{\alpha \frac{1}{2\alpha}\hat{K}_T}, \; c = (E(e^{\alpha \frac{1}{2\alpha}\hat{K}_T}))^{-1}. \tag{4.17}$$

Consequently, we can confirm, the minimal entropy and the minimal martingale measure coincide, if $\hat{K}_T$ is deterministic (the density is equal to one and so we get the same entropy, so by uniqueness both measures coincide). Moreover, they are the same if $E_Q\hat{K}_T$ does not dependent on Q. Note, (4.16) is equivalent to the following problem (see (3.76) below):

$$\sup_{X\in\mathcal{W}(0)} E(-\exp\{-\alpha(x+X-\xi)\}) \tag{4.18}$$

$$= \quad -\exp\{\alpha \sup_{Q\in\mathbb{P}_f}(E_Q(\xi)-x-\frac{1}{\alpha}H(Q|P))\},\ \xi=\frac{1}{2\alpha}\hat{K}_T \tag{4.19}$$

See section 3.2.2 (Exponential Utility Function), Proposition 3.2.11 or Delbaen et al (2002) Theorem 5.1..

4.1.3 q-Optimal Measure

We next define the $q-$ minimal measure, i.e. we search for a measure which minimizes the qth power of its density. First, we discuss existence and uniqueness results of the $q-$optimal measure. Under very general assumptions Leitner (2001) prove an existence and uniqueness result. The author uses a general result from convex analysis, see Ekeland and Temam (1976), Proposition II.1.2. However, this result only insures that the minimizer Q_q is in the class of all local signed martingale measures, i.e. the corresponding density is q-integrable, but it is not always positive. To guarantee that the density of the minimizer Q_q is non-negative and Q_q equivalent to P, we have to impose stronger assumptions. We show that these assumptions are satisfied in the Brownian Model with bounded coefficients. Afterwards, we define the $q-$optimal measure under these stronger assumption and further simplify the definition. The simple version has the advantage that it can be described by a BSDE in a quite general semimartingale setting, developed by Mania et al (2003a). This approach can be used to give a necessary and sufficient condition for the equivalence of the minimal and the q-optimal measure. Secondly, we consider the special case of $q=2$, i.e. the variance-optimal measure. We give a characterization of it in the Brownian Model via a Ricatti-BSDE. This directly confirms the above assertion: If the mean-variance-process is deterministic, then the variance-optimal measure coincides with the minimal martingale measure. We further extend these results to an arbitrary $q>1$. From above, we know that in this case also the minimal entropy measure coincides with the minimal martingale measure, so all criterions are equivalent in the case of a deterministic mean-variance process at time T. We conclude the section with a simple example in the Brownian Model, where we can directly calculate the q-optimal measure.

In a quite general setting, one can find a existence and uniqueness result of the q-minimal measure in Leitner (2001), Proposition 1.5.11 (Corollary 1.5.12.), here only stated for isoelastic utilities ($-U \in \mathcal{C}^p$, where $U(x) =$

$\frac{-|x|^p}{p}$, $p > 1$, $x \in \mathbb{R}$). Further, the assumption $S \in \mathbb{A}^{(p)}_{loc}$ is satisfied since we work on a finite time interval and S is locally bounded (see Leitner (2001), Preliminaries for notation):

Theorem 4.1.12. *Let $\mathcal{M}^q_S \neq \emptyset$ and $U(x) = -|x|^p$, $p > 1$, $x \in \mathbb{R}$, and $\overline{\mathcal{G}^p(0)} = \overline{\{Y_T : Y \in \mathcal{W}(0)\}}$ then the solution of the optimization problem:*

$$E(U(W^{min})) = \sup_{W \in x + \overline{\mathcal{G}^p(0)}} E(U(W)),\ x \in L^p \backslash \overline{\mathcal{G}^p(0)},\ W_{min} \in x + \overline{\mathcal{G}^p(0)}$$

uniquely exists. Additionally, with $\tilde{y} := E(-U'(W_{min})x) > 0$ the element $\tilde{Z}_q = -U'(W_{min}) \in \tilde{y}\mathcal{M}^q_{S,Z}$, $\frac{1}{p} + \frac{1}{q} = 1$ solves the dual problem:

$$E(\check{U}(\tilde{Z}_q)) = \inf_{\tilde{Z} \in \tilde{y}\mathcal{M}^q_{S,Z}} E(\check{U}(\tilde{Z})),$$

$\tilde{Z}_q$ is unique since U is continuously differentiable. Further, we have for $x = 1$:

$$Z_q = \frac{sgn(W_{min})p|W_{min}|^{p-1}}{\tilde{y}} \in \mathcal{M}^q_{S,Z},\ \tilde{y} = E(sgn(W_{min})p|W_{min}|^{p-1}) > 0$$

and

$$\|W_{min}\|_{L^p}\|Z_q\|_{L^q} = 1,\ \|W_{min}\|_{L^p} \leq 1,\ \|Z_q\|_{L^q} \geq 1.$$

Note, the definition of the Fenchel dual is slightly different from our definition ($\hat{U}(y) := \sup\{U(x) + xy\} = \check{U}(-y)$). The corresponding transformation yields the above result. We are looking for unique measure minimizing $E(Z^q)$ with $q > 1$. Using

$$U(x) = (-x)^{\frac{q}{q-1}}(q^{\frac{-1}{q-1}} + q^{\frac{-q}{q-1}})$$

with $\frac{q}{q-1} > 1$ and U continuously differentiable and choosing x such that $\tilde{y} = 1$, the dual problem becomes: $\inf_{Z \in \mathcal{M}^q_{S,Z}} E(Z^q), \check{U}(Z) = Z^q$. So Z_q is the $q-$optimal measure.

In the setting of Leitner (2001), we define:

Definition 4.1.11. Let $q > 1$, then the $q-$ minimal measure, denoted by Q_q, with density process $Z^{(q)}$ and density $Z_q := Z^{(q)}_T$, is defined as the unique solution of:

$$\min_{Z \in \mathcal{M}^q_{S,Z}} E(Z^q) \tag{4.20}$$

As explained above, the L^p setting fits well, however, in the primal problem they optimize over the closure of all final values of all value processes, which can replicated by a simple self-financing strategy ($\overline{\mathcal{V}^p}$). So there might exist elements which cannot be replicated by our L^p-strategies $\mathcal{A}^p$. Consequently, also the set we optimize over in the dual is different, it is the set of all signed local uniformly integrable martingale measures, but these "measures" can become negative. This is not applicable in our case. However, by Lemma 1.2.3 we have that $\overline{\mathcal{V}}^p = \overline{\mathcal{G}}^p$. So if $\mathcal{G}^p$ -the set of all hedgable L^p-claims- is closed, both problems coincide. Theorem 1.2.4 shows that $\mathcal{G}^p$ is closed under assumption 1.2.4. Proposition 1.5.11 in Leitner (2001) further shows the existence of a unique solution of the dual problem in their setting. This uniqueness can be transferred to our problem, since the dual functional of course stays strictly convex, since our functionals are the same up to a minus sign. So the uniqueness also follows. All necessary properties of the used Proposition II.1.2. in Ekeland and Temam (1976) are satisfied, therefore the maximum of the primal problem uniquely exists. This idea is manifested in Theorem 4.1.13, precisely proved by Grandits and Krawczyk (1998).

To adjust the existence result in Leitner (2001) to our setting, we have to use a assumption proposed in section 1.2.1. We defined the Reverse Hölder inequality $R_p(Q)$, see Definition 1.2.4: A process Z satisfies the Reverse Hölder inequality $R_q(Q)$, if there exists a $C(q) > 1$ such that

$$\sup_{\tau \in \mathcal{T}} E_Q(|\frac{Z_T}{Z_\tau}|^q | F_\tau) < C(q). \tag{4.21}$$

Recall, assumption 1.2.4 :

1. All $(F, P)-$ local martingales are continuous (A).

2. There exists an equivalent martingale measure Q, its density respectively which satisfies the reverse Hölder inequality $R_{p_0}(P)$ for some fixed $p_0 > 1$ (B).

Note again, these assumptions imply that $\mathcal{M}_q^e \neq \emptyset$. We again state the Theorem 4.1 proved in Grandits and Krawczyk (1998) (extending Theorem 1.2.4). It also includes important information about the q-optimal measure:

Theorem 4.1.13. *Let S be a continuous semimartingale, $1 < p < \infty$, and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following assertion are equivalent:*

1. *There exists a martingale measure Q in $\mathcal{M}_e^q$ and $\mathcal{G}^p(0)$ is closed in $L^p(P)$.*

2. *There exists a martingale measure Q in $\mathcal{M}_e^q$ that satisfies the Reverse Hölder inequality $R_q(P)$*

3. *The q-optimal martingale measure Q_q is in $\mathcal{M}_e^q(S)$ and satisfies $R_q(P)$*

So we can define under assumption A and B (it suffices that S is continuous):

Definition 4.1.12. Let $q > 1$, then the $q-$ minimal measure, denoted by Q_q and its density process by $Z^{(q)}$ or its density by $Z_q := Z_T^{(q)}$, is defined as the unique solution of:

$$\min_{Z \in \mathcal{M}^q_{e,Z}} E(Z^q) \tag{4.22}$$

In the Brownian model with bounded coefficients, we can use definition 4.1.12 without loss of generality:

Example 4.1.1. In the Brownian Model with bounded coefficients assumption A is satisfied by definition. The Reverse Hölder is satisfied for the minimal martingale measure, i.e. the density is generated by $\mathcal{E}_T(-\int \theta' dW_s)$:

$$\begin{aligned}
\left|\frac{Z_T^q}{Z_\tau^q}\right| &= (\exp\{-\int_\tau^T \theta_s' dW_s - \frac{1}{2}\int_\tau^T \|\theta_s\|^2 ds\})^q \\
&= \exp\{-\int_\tau^T q\theta_s' dW_s - \frac{1}{2q}\int_\tau^T q^2\|\theta_s\|^2 ds\} \\
&= \exp\{-\int_\tau^T q\theta_s' dW_s - \frac{1}{2}\int_\tau^T q^2\|\theta_s\|^2 ds + \frac{q-1}{2q}\int_\tau^T q^2\|\theta_s\|^2 ds\} \\
&\le C_T(q)\exp\{-\int_\tau^T q\theta_s' dW_s - \frac{1}{2}\int_\tau^T q^2\|\theta_s\|^2 ds\}
\end{aligned}$$

$C(q) \in [1, \infty)$ a constant, depending on θ, since θ is bounded. Further, Novikov's condition is satisfied, thus

$$\tilde{Z}_t^q := \exp\{-\int_0^t q\theta_s' dW_s - \frac{1}{2}\int_0^t q^2\|\theta_s\|^2 ds\}$$

is a martingale, hence:

$$\begin{aligned}
& E\left(\left|\frac{Z_T^q}{Z_\tau^q}\right| |F_\tau\right) \\
\le\ & E\left(C_T(q)\exp\{-\int_\tau^T q\theta_s' dW_s - \frac{1}{2}\int_\tau^T q^2\|\theta_s\|^2 ds\}\middle| F_\tau\right) \\
\le\ & C_T(q)\frac{E\left(\exp\{-\int_0^T q\theta_s' dW_s - \frac{1}{2}\int_0^T q^2\|\theta_s\|^2 ds\}|F_\tau\right)}{\exp\{-\int_0^\tau q\theta_s' dW_s - \frac{1}{2}\int_0^\tau q^2\|\theta_s\|^2 ds\}} = C_T(q)
\end{aligned}$$

■

Another existence and uniqueness result in the Brownian Model, not using the Reverse Hölder inequality, can be found in Bürkel (2004), Theorem 6.7 and its subsequent discussion. The author uses a result from convex analysis taken from Luenberger (1969), (§5.8 Theorem 1 with the appropriate spaces). More generally, the Reverse Hölder inequality is satisfied for the minimal martingale measure when the final value of the mean-variance-process is bounded:

Example 4.1.2. Suppose there is a $C \in (0, \infty)$ such that $\int_0^T \hat{\lambda}'_s d\langle M\rangle_s \hat{\lambda}_s \leq C$, then

$$\begin{aligned}
\left|\frac{Z_T^q}{Z_\tau^q}\right| &= \exp\{-\int_\tau^T q\hat{\lambda}'_s dM_s - \frac{1}{2q}\int_\tau^T q^2 \hat{\lambda}'_s d\langle M\rangle s \hat{\lambda}_s\} \\
&= \exp\{-\int_\tau^T q\hat{\lambda}'_s dM_s - \frac{1}{2}\int_\tau^T q^2 \hat{\lambda}'_s d\langle M\rangle_s \hat{\lambda}_s \\
&\quad + \frac{q-1}{2q}\int_\tau^T q^2 \hat{\lambda}'_s d\langle M\rangle_s \hat{\lambda}_s\} \\
&\leq \exp\{-\int_\tau^T q\hat{\lambda}'_s dM_s - \frac{1}{2}\int_\tau^T q^2 \hat{\lambda}'_s d\langle M\rangle_s \hat{\lambda}_s \\
&\quad + \frac{q-1}{2q} q^2 \int_0^T \hat{\lambda}'_s d\langle M\rangle_s \hat{\lambda}_s\}
\end{aligned}$$

An analogous calculation as in the Brownian example yields the Reverse Hölder inequality in the case of a bounded final value of the mean-variance-process. Note, if $\langle -\int \hat{\lambda}' dM\rangle$ is bounded then also $\langle -\int q\hat{\lambda}' dM\rangle$. So $\mathcal{E}(-\int q\hat{\lambda}' dM)$ is also a martingale. For later chapters, we set $\tau = 0$ and obtain:

$$C_T(q) = \exp(\frac{q(q-1)}{2}\langle \int \hat{\lambda}' dM\rangle).$$

$$C_T(q) \leq \exp(\langle \int \hat{\lambda}' dM\rangle),\ q \in [1,2]$$

So

$$\begin{aligned}
&E\left(\left|Z_T^q\right|\right) \\
\leq\ & C_T(q) \leq \exp(\langle \int \hat{\lambda}' dM\rangle_T),\ q \in [1,2]
\end{aligned} \tag{4.23}$$

■

By Theorem 2.2.6, we can further reformulate definition 4.1.12:

$$\inf_{Q \in \mathcal{M}_e^q} E((\mathcal{E}_T(M^Q))^q),\ q > 1 \tag{4.24}$$

where M^Q a P-local martingale such that $Z_Q = \mathcal{E}(M^Q)$. The last reformulation has the advantage that it can be described by a backward stochastic differential equation (BSDE), also in the semimartingale case (see Mania et al (2003a)). An explicit solution of (4.24) is obtained in some special cases (Theorem 1, setting $\eta = 1$, here Theorem 4.1.14):

Recall, $\mathcal{N}$ is the class of all local martingales orthogonal to M such that $\mathcal{E}_t(-\int \hat{\lambda}' dM + N)$, $t \in [0,T]$ is a martingale under P and $\mathcal{N}_q$ a subclass where $\mathcal{E}_t(-\int \hat{\lambda}' dM + N)$, $t \in [0,T]$ is a strictly positive $P-$martingale with $E\mathcal{E}_t^q(-\int \hat{\lambda}' dM + N) < \infty$, $t \in [0,T]$ in addition. So we have:

Theorem 4.1.14. *If $\mathcal{M}_q^e \neq \emptyset$ and all $P-$local martingales are continuous, then the following assertions are equivalent:*

1. *the martingale measure Q_q is q-minimal*
2. *Q_q is a measure satisfying*

$$dQ_p = \mathcal{E}_T(M^{Q_p})dP, \tag{4.25}$$

where

$$M_t^{Q_q} = -\int_0^t \hat{\lambda}_s' dM_s - \frac{1}{q-1}\int_0^t \frac{1}{V_s(q)} d\tilde{m}_s(q) \tag{4.26}$$

$V(q) = V_0 + m(q) + A(q)$ is the value process of problem 4.24, it uniquely solves the following BSDE:

$$\begin{aligned} Y_t &= Y_0 - \operatorname{ess}\inf_{N\in\mathcal{N}_q}[\frac{1}{2}q(q-1)\int_0^t Y_s d\langle -\int \hat{\lambda}' dM + N\rangle_s \\ &\quad + q\langle -\int \hat{\lambda}' dM + N, L\rangle_t] + L_t,\ t<T, Y_T = 1. \end{aligned}$$

$\tilde{m}(q)$ denotes the orthogonal part of the G-K-W-decomposition of $m(q)$:

$$m_t(q) = \int_0^t \phi_s'(q) dM_s + \tilde{m}_t(q) \tag{4.27}$$

If $\mathcal{E}(-\int_0^t \hat{\lambda}_s' dM_s)$ is martingale measure, i.e. the minimal martingale measure exists and in addition it satisfies the Reverse Hölder condition, then the value process $V(q)$ above is a unique solution of the above BSDE and there exist positive constants c and C such that for all $t \in [0,T]$:

$$c \leq V_t(q) \leq C \quad P-a.s.$$

Further, by Mania et al (2005) Proposition 2 we have for the martingale part $m(q)$ of $V(q)$:

Proposition 4.1.15. *Let assumption 1.2.4 be satisfied then $m(p) \in BMO$ and for some positive constant $C > 0$:*

$$\|m(p)\|_{BMO} \leq C(q-1)^2.$$

Note, further under assumption 1.2.4, Mania et al (2005) (Proposition 3) prove that:

$$E(\langle \frac{m(q)}{q-1} - m\rangle_T) \to 0,\ q \downarrow 1$$

where m is defined in Theorem 4.1.10. Using this they prove L^1-convergence of the q-optimal measures to the minimal entropy measure, see Theorem 4.2.1 or chapter 5 for a proof.

By Theorem 4.1.14 it is not surprising that in Schweizer (1995) (Theorem 7) the following connection to the minimal martingale measure is proven:

Theorem 4.1.16. *Suppose S is continuous and there exists an equivalent martingale measure. If $\hat{K}_T$ is deterministic, then the minimal martingale measure coincides with the variance-optimal measure (2-minimal measure), i.e. Q_{mmm} is the unique solution of:*

$$\min_{Q\in\mathcal{M}_e^2(P)} \sqrt{Var(\frac{dQ}{dP})} \Leftrightarrow \min_{Q\in\mathcal{M}_e^2(P)} E\left(\left(\frac{dQ}{dP}\right)^2\right) \tag{4.28}$$

This corresponds to the following result:

Theorem 4.1.17. *In the Brownian case with bounded coefficients, the Variance-optimal-measure has the following form:*

$$\frac{dP_V}{dP} = \mathcal{E}_T(-\int_0^T \tilde{\lambda}' dW_s) \tag{4.29}$$

where

$$\tilde{\lambda} = \bar{\theta} - [I - \sigma'(\sigma\sigma')^{-1}\sigma]LK^{-1}$$

(K, L) denotes the solution pair of the Riccati BSDE associated to the mean-variance control problem ($U(x) = -x^2$) stated in Kohlmann (2003) p.163/164.

L is zero if the coefficients are deterministic see page 142 in Kohlmann (2003). Hence, the variance-optimal measure coincides with the minimal martingale measure which itself matches with the minimal entropy measure, again since $\hat{K}_T$ is deterministic. This observation is extendable. Bürkel (2004) (above all chapter 6) has derived the solution of the $q-$minimal measure as a function of the solution of the corresponding Riccati-BSDE (K, L). It does not depend on q, if L is zero, e.g. if the coefficients are deterministic. The $q-$minimal measure coincides with the variance-optimal and which again is the minimal martingale and minimal entropy measure.

In fact, $q-$minimal measure does not depend on q if $\hat{K}_T$ is deterministic. As we will derive next. First we summarize our findings, if $\hat{K}_T$ is deterministic, then the minimal entropy coincides with the minimal martingale measure and the variance-optimal measure and finally all $q-$minimal measures are independent of $q > 1$. This already shows that the q-minimal measure exists and equals the minimal martingale measure, so in the Brownian case it is the same as $\mathcal{E}_T(-\int_0^T \bar{\theta}_s dW_s)$.

Theorem 4.1.18. *Suppose S is continuous and ($\mathcal{F}_t$-adapted) and $\mathcal{M}_e^q \neq \emptyset$, $q > 1$. If $\hat{K}_T$ is deterministic, then the minimal martingale measure is the unique solution of:*

$$\min_{Q\in\mathcal{M}_e^q} E\left(\left(\frac{dQ}{dP}\right)^q\right) \tag{4.30}$$

Proof. We assume that $\hat{K}_T = \langle -\int \hat{\lambda}' dM \rangle_T = \int_0^T \hat{\lambda}' d\langle M \rangle \hat{\lambda}$ is deterministic. Hence, by Novikov's condition $\hat{Z}_q = \mathcal{E}(-\int_0^T q\hat{\lambda}' dM)$ is a martingale for every $q \in \mathbb{R}$ and therefore $E(\hat{Z}_q) = 1$ and for $q > 1$, we have that $\hat{Z}_q \in L^m(P)$ for every $m \geq 1$:

$$\begin{aligned}
\hat{Z}^q &= \exp\{-\int_0^T q\hat{\lambda}'_s dM_s - \frac{1}{2q}\int_0^T q^2 \hat{\lambda}'_s d\langle M \rangle s \hat{\lambda}_s\} \\
&= \exp\{-\int_0^T q\hat{\lambda}'_s dM_s - \frac{1}{2}\int_0^T q^2 \hat{\lambda}'_s d\langle M \rangle_s \hat{\lambda}_s \\
&\quad + \frac{q-1}{2q}\int_0^T q^2 \hat{\lambda}'_s d\langle M \rangle_s \hat{\lambda}_s\}
\end{aligned}$$

Further,

$$\begin{aligned}
\check{Z} &:= \exp\{-\int_0^T (q\hat{\lambda}_s)' dM_s - \frac{1}{2}\int_0^T (q\hat{\lambda}_s)' d\langle M \rangle_s (q\hat{\lambda}_s) \\
&= \mathcal{E}(-\int^T q\hat{\lambda}' dM) && (4.31) \\
&\Rightarrow E(\check{Z}) = 1 && (4.32)
\end{aligned}$$

Finally,

$$\begin{aligned}
E(\hat{Z}^q) &= E(\exp\{-\int_0^T q\hat{\lambda}'_s dM_s - \frac{1}{2q}\int_0^T q^2 \hat{\lambda}'_s d\langle M \rangle s \hat{\lambda}_s\}) \\
&= E(\exp\{-\int_0^T q\hat{\lambda}'_s dM_s - \frac{1}{2}\int_0^T q^2 \hat{\lambda}'_s d\langle M \rangle_s \hat{\lambda}_s \\
&\quad + \frac{q-1}{2q}\int_0^T q^2 \hat{\lambda}'_s d\langle M \rangle_s \hat{\lambda}_s\}) \\
&= E(\exp\{-\int_0^T q\hat{\lambda}'_s dM_s - \frac{1}{2}\int_0^T q^2 \hat{\lambda}'_s d\langle M \rangle_s \hat{\lambda}_s\}) \\
&\quad \times \exp\{\frac{q-1}{2q}\int_0^T q^2 \hat{\lambda}'_s d\langle M \rangle_s \hat{\lambda}_s\} \\
&= \exp\{\frac{q^2(q-1)}{2q}\int_0^T \hat{\lambda}'_s d\langle M \rangle_s \hat{\lambda}_s\} \\
&= \exp\{\int_0^T \hat{\lambda}'_s d\langle M \rangle_s \hat{\lambda}_s\}^{\frac{q^2(q-1)}{2q}} && (4.33)
\end{aligned}$$

We know that if $\mathcal{M}_e^q$ is not empty than the minimizer is already equivalent. For an arbitrary $Z_T \in \mathcal{M}_{e,Z}^q$, we know from Theorem 2.2.6 that:

$$Z = \mathcal{E}(-\int \hat{\lambda}' dM)\mathcal{E}(L)$$

So analogously, we have:

$$\begin{aligned} Z_T^q &= \mathcal{E}_T\{-\int q\hat{\lambda}_s' dM_s + qL\} \\ &\quad \times \exp\{\frac{q-1}{2q}\int_0^T q^2\hat{\lambda}_s' d\langle M\rangle_s\hat{\lambda}_s\}\exp\{\frac{(q-1)q}{2}[qL]\} \\ &\geq \mathcal{E}_T\{-\int q\hat{\lambda}_s' dM_s + qL\}\exp\{\frac{q-1}{2q}\int_0^T q^2\hat{\lambda}_s' d\langle M\rangle_s\hat{\lambda}_s\} \end{aligned}$$

Thus,

$$E(Z_T^q) \geq E(\mathcal{E}_T(-\int q\hat{\lambda}_s' dM_s + qL))\exp\{\frac{q-1}{2q}\int_0^T q^2\hat{\lambda}_s' d\langle M\rangle_s\hat{\lambda}_s\}.$$

A straightforward generalization of the results in Heyde and Wong (2004):

$$E(\mathcal{E}_T\{-\int q\hat{\lambda}_s' dM_s + qL\}) = 1.$$

Thus,

$$E(Z_T^q) \geq E(\hat{Z}_T^q).$$

for every q and $Z_T \in \mathcal{M}_{e,Z}^q$. □

In the Brownian case with deterministic coefficients, we can use a slightly different proof:

Proof. By Lemma 1.2.5 every density can be represented by $\mathcal{E}(-\int \theta dW_s)$, where θ is deterministic, because σ and μ are deterministic and so $\langle -\int \theta dW_s\rangle$. We have as in (4.33):

$$\begin{aligned} E(Z_T^q) &= E(\exp\{-\int_0^T q\theta_s' dW_s - \frac{1}{2}\int_0^T q^2\theta_s' ds\theta_s\}) \\ &\quad \times \exp\{\frac{q-1}{2q}\int_0^T q^2\theta_s' ds\theta_s\} \\ &= \exp\{\frac{q-1}{2q}\int_0^T q^2\theta_s' ds\theta_s\} \end{aligned}$$

We have already seen the set over which we minimize is the same for all q. But the minimizer of $(\exp\{\int_0^T \theta' ds\theta_s\}^{\frac{q^2(q-1)}{2q}})$ is also the same as of $(\exp\{\int_0^T \theta' ds\theta_s\}$ (monotone transformation). The assertion follows, since the solution for $q = 2$ exists and is equal to the minimal martingale measure due to Schweizer (1995). □

Theorem 4.1.18 is also done in Mania et al (2003a), Corollary 3 and the subsequent remark, using a different approach. Using the representation (4.26) in Theorem 4.1.14, they give necessary and sufficient condition when the minimal martingale measure coincides with a specific q-optimal measure.

Theorem 4.1.19. *The minimal martingale measure is q-optimal if and only if:*

$$\mathcal{E}_T^q(-\int^T \hat{\lambda}' dM) = c + \int^T g_s' dM_s$$

for some M-integrable predictable g and the process ($\int_0^t g_u' dS_u, t \in [0,T]$) is a $Q_{mmm}-$martingale.

Further, it holds:

$$\begin{aligned}\mathcal{E}_T^q(-\int^T \hat{\lambda}' dM) &= \mathcal{E}_T(-q\int^T \hat{\lambda}' dM)\exp(\frac{q(q-1)}{2}\langle\int \hat{\lambda}' dM\rangle_T) \\ &= \exp(\frac{q(q-1)}{2}\langle\int \hat{\lambda}' dM\rangle_T) \\ &\quad \times(1-p\int_0^T \mathcal{E}_s(-\int^s \hat{\lambda}' dM)\hat{\lambda}_s' dM_s)\end{aligned}$$

The last equality is true since $\mathcal{E}_T^q(-\int \hat{\lambda}_s' dM_s)$ satisfies the Doléans-Dade-equation. The first term is constant, if $\langle -\int_0^\cdot \hat{\lambda}_s' dM_s\rangle_T$ is deterministic. That holds for every q.

Theorem 4.1.18 is even more obvious, in the Brownian setting if the coefficients are constant in addition, we have:

$$\exp\{\int_0^T \hat{\lambda}_s' d\langle M\rangle_s \hat{\lambda}_s\} = \exp\{T\theta'\theta\},$$

hence

$$\min\ \exp\{\theta'\theta T\} \text{ s.t. } \sigma\theta = \mu \tag{4.34}$$

The necessary integrability and martingale conditions are satisfied since the coefficients are constant. The corresponding Lagrange function is:

$$\mathcal{L}(\theta,\beta) = \exp\{\theta'\theta T\} - (\sigma\theta - \mu)\beta \tag{4.35}$$

e^x is convex, so we just have to consider the derivative:

$$\frac{\mathcal{L}(\theta,\beta)}{d\theta} = 2\theta\frac{1}{(2m-1)^2}\exp\{\theta'\theta T\} - \sigma\beta \tag{4.36}$$

which is zero if and only if $\theta \in im\ (\sigma)$. So the minimizer of (4.34) is the minimal martingale measure!

4.2 Convergence of the q-Optimal to the Minimal Entropy Measure

We have suggested several measures. It arose the question, when these measures are equal or not, we presented some selected results. In this section, we give some references. The main goal of this section is therefore to show that the $q-$optimal measure converges to the minimal entropy measure. The kind of convergence depends on the strength of our assumptions. We mainly state a result from Mania et al (2005).

Almost sure convergence is clear if the coefficients are deterministic (or even in a semimartingale model if $\hat{K}_T$ is deterministic), then all $q-$optimal measures coincide with the minimal entropy measure, see Theorem 4.1.18. The sequence is almost surely constant. So trivially: $Z_q \overset{a.s}{\to} Z_{min}$.

In the non-deterministic case, usually we do not have that $Z_{2m} = Z_{min} = Z_{mmm}$. However, there exists a relationship between the q-optimal measures and the minimal entropy measure. One is worked out in Mania et al (2005). The density process of the q-minimal measure converges to one of the minimal entropy measure in $\mathcal{H}^1$, and so their densities in L^1, for $q \downarrow 1$.

We start with some reference when $Z_q = Z_{min} = Z_{mmm}$ is not satisfied. In Mania and Tevzadze (2003), necessary and sufficient conditions of the equality of the variance optimal measure and the minimal martingale measure are given. Mania et al (2003b) provide the reader with necessary and sufficient conditions when the entropy measure does coincide with the minimal martingale measure. Under some conditions that is, in fact, the case if and only if $\hat{K}_T$ is deterministic, see Theorem 4.1.11. Similar conditions for the q-optimal measure can be found in Mania et al (2003a), see Theorem 4.1.19. In these cases, we want to know when Z_q converges to density of the minimal entropy measure. Grandits and Rheinländer (2002a) prove the convergence in entropy. The approach uses results from duality theory. Later Mania et al (2005) prove the convergence in $\mathcal{H}^1$ with a BSDE-approach. Then the densities converge in L^1, which implies the convergence in probability. The approach does not use any duality theory and it only focuses on the density process of the q-optimal measure as a solution of a BSDE. So not to go around in circles, when using duality relationships in section 5.1, we then rely on the latter more general result:

Theorem 4.2.1. *Suppose that assumption 1.2.4 is satisfied. Then, the density process* $Z_t^{(q)} = \mathcal{E}_t(M^{Q_q})$ *of the q-minimal measure* Q_q *converges to density process* $Z_t^{min} = \mathcal{E}_t(M^{Q_{min}})$ *of the minimal entropy measure* Q_{min} *in* $\mathcal{H}_1$*, i.e.*

$$E \sup_{t \leq T} |Z_t^{(q)} - Z_t^{min}| \to 0, \; q \downarrow 1,$$

in particular:

$$Z_T^{(q)} \overset{L^1}{\to} Z_T^{min}, \; q \downarrow 1.$$

Note, the notation is the same as in Theorem 4.1.14. Further, $Z^{(q)}$ is density process and therefore at least a local martingale. By Burkholder-Davis-Gundy inequality $\mathcal{S}^1$ and $\mathcal{H}^1$ are equal.

We can show more:

Lemma 4.2.2. *Under assumption 4.2.1 or assumption 1.2.4* $Z_q = Z_T^{(q)}$ *converges to* Z_{min} *in* $L^{1+\mu}$*, i.e.:*

$$E(||Z_q - Z_{min}|^{1+\mu}|) \to 0,$$

where μ *is chosen as in (5.25).*

Proof. We know that $|Z_q - Z_{min}| \xrightarrow{P} 0$ and therefore $|Z_q - Z_{min}|^{1+\mu} \xrightarrow{P} 0$, since $|Z_q - Z_{min}|^{1+\mu}$ is uniformly integrable by Theorem 5.2.3 (see Chapter 5 proof of item 1a non-deterministic case), from convergence in probability follows convergence in L^1. Hence $E(||Z_q - Z_{min}|^{1+\mu}|) \to 0$. □

The theorem in Mania et al (2005) is a generalization of a result of Grandits and Rheinländer (2002a) with a completely different approach. They even impose a stricter assumption (it implies 1.2.4B when considering the sequence of q-optimal measures, see below):

Assumption 4.2.1. *The considered semimartingale* S *is continuous and there exists a* $Q \in \mathcal{M}^e_{q_0}(P)$ *for a* $q_0 > 1$ *and its density process* Z *satisfies the Log-Reverse Hölder condition* ($R_{LLogL}(P)$) *with constant* K*, i.e. there exists a* $K > 0$ *such that*

$$\sup_{\tau \in \mathcal{T}} E_P\left(\frac{Z_T}{Z_\tau} \log^+ \frac{Z_T}{Z_\tau} | F_\tau\right) < K. \tag{4.37}$$

$\mathcal{T}$ = the set of all stopping times

and

Assumption 4.2.2. *A positive process* Z *satisfies condition (S) if there exists a constant* $C > 0$ *such that:*

$$\frac{1}{C} Z_- \leq Z \leq C Z_-$$

The density process of the minimal entropy measure and the q-optimal measures satisfy condition (S), if assumption 4.2.1 holds (see Lemma 4.6. and Lemma 4.10. Grandits and Rheinländer (2002a)). They further show the following lemma (Lemma 2.2.):

Lemma 4.2.3. *If* Z *satisfies assumption 4.2.1 with constant* K_l *and condition (S) with constant* C_l*, then there exist constants* J *and* K_q *only depending on* K_l *and* C_l *such that:*

$$M = \int \frac{dZ}{Z_-} \in BMO(P)$$

Hence, $(Z_t)_t = (\mathcal{E}_t(M))_t$ *is a uniformly integrable martingale and* Z *satisfies the* $R_q(P)$ *for a* $q > 1$ *with constant* K_p *(implying assumption 1.2.4B).*

Since the minimal and the q-optimal measures both satisfy condition (S) under assumption 4.2.1, assumption 4.2.1 implies assumption 1.2.4.B. Note, since $M \in BMO$, we automatically have that $\mathcal{E}(M)$ is a uniformly integrable martingale.

We next state the result from Grandits and Rheinländer (2002a). Before, we define convergence in entropy (implying convergence in L^1):

Definition 4.2.1. A sequence $(Q_n)_n$ of probability measures converges in entropy to a probability measure Q, if

$$\lim_{n\to\infty} H(Q_n, Q) = 0$$

Theorem 4.15. in Grandits and Rheinländer (2002a) prove the following assertion:

Theorem 4.2.4. *Under assumption 4.2.1,* $(Q_q)_{q\geq 1}$ *denoting the sequence of* $q-$*optimal measures converges in entropy to the minimal entropy measure* Q_{min}.

Recall, H is not a metric, therefore a surprising result!

Chapter 5

Convergence to the Exponential Problem

In this chapter, we introduce a sequence of utility functions such that the solution of the corresponding utility problem converges to the one of the exponential utility problem. We apply the results from the previous chapter to the exponential problem via an approximation approach. Firstly, we introduce the approximating sequence and discuss the existence and uniqueness of the corresponding optimization problem. Secondly, we tackle the approximation of the solution of the exponential problem by the solutions of introduced sequence using the results stated in section 4.2.

5.1 Approximating Sequence of the Exponential Utility Function

In this section, we do not consider the partial sums of the exponential utility function. We rather analyze another sequence approximating $-e^{-x}$ and solve its optimization problem. We define:

$$u_{2m}(x) = -(1 - \frac{x}{2m})^{2m}$$

We derive the Fenchel-duals to u_{2m} and show that they converge to $-y + y \log y$, i.e. to the Fenchel-dual of $-e^{-x}$. The solution of the dual problem with index $2m$ is the $\frac{2m}{2m-1}$−optimal measure. The solution of the primal problem can then be represented as a function of the $\frac{2m}{2m-1}$−optimal measure. The $\frac{2m}{2m-1}$−optimal measure does not depend on the initial wealth, see equation 5.5 below. So again we can explicitly solve $\mathcal{X}^{2m}_{Z^{2m}(y)}(y) = x$ easily since it is equivalent to $\mathcal{X}^{2m}_{Z^{2m}}(y) = x$ to get the "y"-part of the dual solution. Recall, $\mathcal{X}^{2m}_{Z^{2m}}(y) = E(Z_{2m}(y)I_{2m}(Z_{2m}y)y)$. Consequently, we can solve the primal problem using the method established in chapter 3. This solution exists and is unique as well. We start with a more theoretical consideration. Since $-(1-\frac{x}{2m})^{2m}$ is continuously differentiable (so its dual is strictly convex), the solution of the dual exists and is unique by slight modifications of the proof of Leitner (2001) Proposition 1.5.11 (see Theorem 5.1.1), also see Theorem 6.7 in Bürkel (2004).

We start modifying Prop. 1.5.11 in Leitner (2001), which gives us the existence and uniqueness of the optimization problems of our sequence:

Theorem 5.1.1. *Let $p = 2r$, $r \in \mathbb{N}$. Under assumption 1.2.4 we have that $\overline{\mathcal{G}^p(0)} = \{Y_T : Y \in \mathcal{W}(0)\} = \mathcal{G}^p(0)$. Consequently, the solution of the optimization problem:*

$$E(U(W^{min})) = \sup_{W \in x+\mathcal{G}^p(0)} E(U(W)),$$

$$\text{where} \qquad x \in L^p \backslash \mathcal{G}^p(0),\ W_{min} \in x + \mathcal{G}^p(0),\ U(\tilde{x}) = -\tilde{x}^p$$

uniquely exists. Additionally, with $\tilde{y} := -E(U'(W_{min})x) > 0$ the element $\tilde{Z}_q = -U'(W_{min}) \in \tilde{y}\mathcal{M}^q_{S,Z}$, $\frac{1}{p}+\frac{1}{q}=1$ solves the dual problem:

$$E(\check{U}(\tilde{Z}_q)) = \inf_{\tilde{Z} \in \tilde{y}\mathcal{M}^q_{S,Z}} E(\check{U}(\tilde{Z})), \tag{5.1}$$

$\tilde{Z}_q$ is unique if U is continuously differentiable. Further, we have for $x = 1$:

$$Z_q = \frac{p(W_{min})^{p-1}}{\tilde{y}} \in \mathcal{M}^q_{S,Z},\ \tilde{y} = E(pW^{p-1}_{min}) > 0$$

and

$$\|W_{min}\|_{L^p}\|Z_q\|_{L^q} = 1,\ \|W_{min}\|_{L^p} \leq 1,\ \|Z_q\|_{L^q} \geq 1. \tag{5.2}$$

Proof. (Sketch)
By Theorem 4.1.13 the first assertion follows. The rest of the proof is the same as in Leitner (2001). Note, the author uses a slightly different convex dual $\hat{U}(y) := \sup\{U(x) + xy\} = \check{U}(-y)$. The corresponding transformation yields the above result. □

By scaling the solution with $-2m$ and adding 1, we obtain the desired existence and uniqueness result. We do not go into detail, since we calculate the solution explicitly using the approach introduced in section 3.2:

Firstly, we derive the convex dual of u_{2m} as defined in (3.17):

$$\check{U}(y) \equiv \sup_{x \in \mathbb{R}}[U(x) - xy] = U(I(y)) - yI(y).$$

The derivative of u_{2m} is $(1-\frac{x}{2m})^{2m-1}$. Hence, we have:

$$I_{2m}(y) := (u')^{-1}_{2m}(y) = (1 - y^{\frac{1}{2m-1}})2m \tag{5.3}$$

Further, the convex dual is:

$$\begin{aligned}\breve{u}_{2m}(y) &= -(1-(1-y^{\frac{1}{2m-1}}))^{2m} - 2m(1-y^{\frac{1}{2m-1}})y \\ &= -y2m(1-y^{\frac{1}{2m-1}}) - y^{\frac{2m}{2m-1}} \\ &\to y\log y - y\end{aligned}$$

since $-n(1-y^{\frac{1}{n}}) \to y\log y$. Next, we consider the dual problem:

$$\min_{y\geq 0,\ Q\in\mathcal{M}_a^q} \phi_{2m}(y,Z) \tag{5.4}$$

where

$$\phi_{2m}(y,Z) = E(\breve{u}_{2m}(Zy)) = E(u_{2m}(I_{2m}(Zy)) - ZyI_{2m}(Zy)) + xy,\ q = \frac{2m}{2m-1}$$

(5.4) converts to:

$$\min_{y\in\mathbb{R}, Z\in\mathcal{M}_{a,Z}^{\frac{2m}{2m-1}}} (2m-1)y^{\frac{2m}{2m-1}}E(Z^{\frac{2m}{2m-1}}) - 2myE(Z)$$

We start optimizing only over the $Z's$ following the method shown in section 3.2. y and Z can be separated. The minimizer of the following two problems is the same, since the solution of the first problem does not depend on $y \geq 0$:

$$\min_{Z\in\mathcal{M}_{a,Z}^{\frac{2m}{2m-1}}} (2m-1)y^{\frac{2m}{2m-1}}E(Z^{\frac{2m}{2m-1}}) - 2my\cdot 1$$

$$\min_{Z\in\mathcal{M}_{a,Z}^{\frac{2m}{2m-1}}} E(Z^{\frac{2m}{2m-1}}) \tag{5.5}$$

Hence, the solution of the dual problem is the $\frac{2m}{2m-1}$−minimal measure, call it Z_{2m}. It is independent of y. So further applying our method from the preceding sections, we next derive $\mathcal{X}_{Z,2m}$ and $\mathcal{Y}_{Z,2m}$:

$$\begin{aligned} &\mathcal{X}_{Z_{2m},2m}(y) = E(Z_{2m}I_{2m}(yZ_{2m})) \stackrel{!}{=} x \\ \Leftrightarrow\quad & E(Z_{2m}2m(1-y^{\frac{1}{2m-1}}Z_{(2m)}^{\frac{1}{2m-1}})) = x \\ \text{and}\quad & \mathcal{X}_{Z_{2m},2m}(y) = 2m - 2my^{\frac{1}{2m-1}}E(Z_{2m}^{\frac{2m}{2m-1}})\end{aligned}$$

Consequently:

$$\mathcal{Y}_{Z_{(2m)},2m}(x) = \left(\frac{2m-x}{2mE\left(Z_{2m}^{\frac{2m}{2m-1}}\right)}\right)^{2m-1} \tag{5.6}$$

Since, we know thatthe optimal measure, which does not depend on y, we can proceed as in the complete case. We have to check the five conditions given in the preceding sections (see property 3.2.1). Firstly, we show that $\mathcal{X}_{2m}$ is invertible. That is trivial since, we consider an odd root function. $\mathcal{X}_{2m}$ has to be finite that is the case since $Z_{2m} \in \mathcal{M}_{e,Z}^{\frac{2m}{2m-1}}$. Further, the domain of $\mathcal{X}_{2m}$ is the real line. However, if the initial wealth is quite large $\mathcal{Y}_{2m}$ might be negative, $\mathcal{Y}_{2m}$ is positive if and only if $x \leq 2m$. So we have to choose m large enough. Otherwise, we invest so much that we can achieve the maximum and consume the rest. Since we are mainly interested in limit theory, we always can choose a large enough m for every initial wealth. In the case $x \leq 2m$, the solution is therefore of the form:

$$X_0^{(2m)}(x) = 2m - 2m \left(Z_{2m} \left(\frac{2m - x}{2mE\left(Z_{2m}^{\frac{2m}{2m-1}} \right)} \right)^{2m-1} \right)^{\frac{1}{2m-1}} \tag{5.7}$$

Finally, we have to check if X_0 is in L^{2m} ($q = \frac{2m}{2m-1}$, and therefore $p = \frac{q}{q-1} = 2m$). That is clear since

$$|X_0^{(2m)}(x)|^{2m} = C(m)|Z_{2m}|^{\frac{2m}{2m-1}} < \infty \tag{5.8}$$

since $Z_{2m} \in L^q$, where $q = \frac{2m}{2m-1}$ and $C(m)$ a constant depending on m.

Duality is clear by construction:

$$\begin{aligned} & \min_{y \geq 0,\, Z \in \mathcal{M}_{e,Z}^q} \phi(Z, y) \\ = \; & \phi_{2m}(\mathcal{Y}_{2m}(x), Z_{2m}) \\ = \; & E(u_{2m}(I_{2m}(Z_{2m}\mathcal{Y}_{2m}(x))) - Z_{2m}\mathcal{Y}_{2m}(x)I_{2m}(Z_{2m}\mathcal{Y}_{2m}(x))) + x\mathcal{Y}_{2m}(x) \\ = \; & \max_{X \in L^p(P): \forall Z: E(ZX) \leq x} E(u_{2m}(X)) \end{aligned}$$

and $E(Z_{2m}I_{2m}(Z_{2m}\mathcal{Y}_{2m}(x))) = x$. So a duality result holds on the mth level and in the limit as seen in section 3.2.2 (Exponential Utility Function).

5.2 Approximation of the Solution of the Exponential Problem

This section presents the proof of the main result of this thesis: The convergence of the solution of the 2m-th problem to the solution of the exponential one. We start with some general considerations and then divide the proof in two parts. We separately treat the deterministic and the non-deterministic case. We start with a short outline of the proofs and the following convention:

Remark 5.2.1. (Convention) The solution of the dual problem is a product of a measure and a non-negative number $Z \cdot y$. For the exponential problem as well as for its approximating sequence, the measure is independent of y. The optimal y is straight forward to calculate, when we already know the optimal Z part. So we usually say the solution of the dual problem and mean the density of the optimal measure Z_{min} or Z_{2m}.

The goal of section 4.2 was to present results that the q-optimal measures converge to the minimal entropy measure, if q tends to 1. This is clear if the coefficients are deterministic (or even in a semimartingale model if $\hat{K}_T$ is deterministic), then all $\frac{2m}{2m-1}-$minimal measures coincide with the minimal entropy measure. Among other things ($\mathcal{Y}_{2m}$, I_{2m} converges to $\mathcal{Y}_{exp}, I_{exp}$ respectively) this yields that the optimal solution of the 2m-th primal problem $X_0^{(2m)}(x) = I_{2m}(Z_{2m}\mathcal{Y}_{2m}(x)) = I_{2m}(Z_{min}\mathcal{Y}_{2m}(x))$ converges almost surely to the solution of the exponential problem: $X_0^{exp}(x) = I_{exp}(Z_{min}\mathcal{Y}_{exp}(x))$.

In the non-deterministic case, we apply a result from Mania et al (2005) (see here Theorem 4.2.1). The density of the $\frac{2m}{2m-1}$-minimal measure converges to the density of the minimal entropy measure in L^1 or more general the corresponding density processes converge in $\mathcal{H}^1$. Unfortunately, products of two L^1-convergent sequences are not necessary convergent. We therefore work with convergence in probability and show that the considered sequences are uniformly integrable, which yields convergence in L^1. To show uniformly integrability, we have to impose more conditions on the set of equivalent martingale measures ((Log)-Reverse-Hölder-condition or weaker assumption 1.2.4). In particular, this condition yields that $Z_{2m}I_{2m}(Z_{2m}y_m)$ and $I_{2m}(Z_{2m}y_m)$ are uniformly integrable. We further show that $Z_{2m}I_{2m}(Z_{2m}y)$ converges to $Z_{min}I_{exp}(Z_{min}y)$ in probability. This and the uniformly integrability (in order to change the order integration and taking limits) implies convergence of $\mathcal{Y}_{2m}$ to $\mathcal{Y}_{exp}$. Furthermore, by the fact that q-minimal measures are all equivalent ($Z_\cdot > 0$) and $Z_{2m} \stackrel{L^1}{\rightarrow} Z_{min}$, we have that $I_{2m}(Z_{2m}\mathcal{Y}_{2m}(x))$ converges to $I_{exp}(Z_{min}\mathcal{Y}_{exp}(x))$ in probability. On account of the uniformly integrability of this sequence, we obtain convergence in L^1 - the desired result.

We started from the problem to maximize an expected utility of a terminal value, so L^1-convergence seems to be appropriate. It follows convergence of an a.s. subsequence. In order to show a.s. convergence for the whole sequence of the primal problem, we need almost sure convergence of the dual solutions. This is till an open question. However, in contrast to Mania et al (2005), we do not consider every sequence $q_m \downarrow 1$, we just look at $q_m = \frac{2m}{2m-1}$. So under certain assumptions P almost sure convergence can be expected.

Finally note, in the deterministic and non-deterministic case for a large initial wealth we have to choose a large enough m in order to obtain a positive $\mathcal{Y}_{2m}$, i.e. our constraint is satisfied with equality.

From the last section, we have:

$$\begin{aligned} X_0^{(2m)}(x) &= I_{2m}(Z_{(2m)}\mathcal{Y}_{(2m)}(x)) \\ I_{2m}(y) &= 2m(1-y^{\frac{1}{2m-1}}) \to -\log y = I_{\exp}(y) \\ \mathcal{X}_{2m}(y) &= E(Z_{(2m)}I_{2m}(yZ_{2m})) \end{aligned}$$

We further need the convergence of the measures, i.e. the convergence of the solutions of the dual problems (Theorem 4.2.1). After some estimations, the convergence of the $\frac{2m}{2m-1}$-minimal measures to the minimal entropy measure yields $Z_{2m}I_{2m}(y_m Z_{2m}) \to Z_{min}I(yZ_{min})$ for an arbitrary real sequence $(y_m)_m$ with limit y. Having this, we show that $\mathcal{Y}_{Z_{2m}(2m)}(x)$ converges to $\mathcal{Y}_{Z_{min},exp}(x)$ or equivalently their corresponding inverse $\mathcal{X}$. Together, this yields:

$$\begin{aligned} X_0^{(2m)}(x) &= I_{2m}(Z_{2m}\mathcal{Y}_{2m}(x)) \\ &\stackrel{P/a.s.}{\longrightarrow} I_{exp}(Z_{min}\mathcal{Y}_{exp}(x)) = X_0^{exp}(x) \end{aligned} \tag{5.9}$$

Note, the convergence depends on the given assumptions and is specified later. Convergence in probability can be strengthened by establishing uniformly integrability of $(I_{2m}(Z_{2m}\mathcal{Y}_{2m}(x)))_m$. Further, to prove that $\mathcal{X}_{Z_{2m}(2m)}(y)$ converges to $\mathcal{X}_{Z_{min},exp}(Y)$, we have to interchange the order of limit and integration in $\lim \mathcal{X}_{2m}(y)$. Apart from the convergence in probability of the integrand $Z_{(2m)}I_{2m}(yZ_{2m})$, we need that $(Z_{2m}I_{2m}(yZ_{2m}))_m$ is dominated by an integrable random variable or is uniformly bounded to apply a dominated convergence result. To use the theorem of dominated convergence (Shiryaev (1996)), it has to be established that the integrand of $\mathcal{X}_{Z_{2m}(2m)}(y)$ is dominated by an integrable random variable independent of m, i.e.

$$|Z_{2m}2m(1-(Z_{2m}y)^{\frac{1}{2m-1}}| \le D \in L^1(P). \tag{5.10}$$

Alternatively, we can show that the integrand is uniformly integrable and bounded in one direction to apply Theorem 4 §6 in Shiryaev (1996):

Theorem 5.2.1. *Let* $0 \le \xi_n \stackrel{P}{\to} \xi$, $E\xi_n < \infty$ *and* $(\xi_n)_n$ *uniformly integrable, then we have:*

$$E\xi_n \to E\xi,\ E|\xi_n - \xi| \to 0 \text{ for } n \to \infty$$

Note, Theorem 4 in Shiryaev (1996) is given for almost sure convergence, but p.262 shows that it is also valid for convergence in probability. Theorem 5 (for almost sure convergence) even shows if $0 \le \xi_n \stackrel{a.s.}{\longrightarrow} \xi$, $E\xi_n < \infty$ that uniformly integrability is necessary and sufficient for $E\xi_n \to E\xi$. The proof exchanges limit and expectation, this is also possible if convergence is only in probability. So Theorem 5 is also valid for convergence in probability:

Theorem 5.2.2. *Let* $(\xi_n) \ge 0$ *a sequence with* $E\xi_n < \infty$ *converging in probability to* ξ. *Then* $E\xi_n \to E\xi$ *if and only if* $(\xi_n)_n$ *is uniformly integrable.*

That means if the series is not uniformly integrable, the solution of the approximating sequence will not converge to the one of the exponential problem.

In our case, the function $x \cdot 2m(1 - x^{\frac{1}{2m-1}})$ is bounded by two from above (see 5.12), so $-x \cdot 2m(1 - x^{\frac{1}{2m-1}}) + 2$ is bigger than or equal to zero. But if $-Z_{2m}2m(1 - (Z_{2m}y)^{\frac{1}{2m-1}} + 2$ converges to a random variable ξ, then $Z_{2m}2m(1 - (Z_{2m}y)^{\frac{1}{2m-1}}$ converges to $-\xi + 2$ in probability (see Shiryaev (1996) p.262), a.s., respectively.

So to obtain the results of this section it remains to impose sufficient conditions such that the following assertion hold:

1. a) $(Z_{2m}2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}))_m$ is dominated by an integrable random variable or alternatively it is uniformly integrable.
   b) $2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}})_m$ is uniformly integrable or dominated by an integrable random variable.

2. $Z_{2m} \overset{L^1/a.s.}{\longrightarrow} Z_{min},\ m \to \infty,\ (Z_{2m} := Z_T^{(\frac{2m}{2m-1})})$

3. With 1 and 2, for every positive, real sequence $(y_m)_m$ with limit y:

$$\begin{aligned} y_m Z_{2m} I_{2m}(Z_{2m}y_m) &= y_m Z_{2m} 2m(1 - (Z_{2m}y_m)^{\frac{1}{2m-1}}) \\ &\overset{L^1/a.s}{\longrightarrow} -yZ_{min}\log(Z_{min}y) = yZ_{min}I_{exp}(Z_{min}y) \end{aligned}$$

4. With 2 and 3, for every positive, real sequence $(y_m)_m$ with limit y:

$$\begin{aligned} & I_{2m}(Z_{2m}y_m) = 2m(1 - (Z_{2m}y_m)^{\frac{1}{2m-1}}) \\ \overset{L^1/a.s.}{\longrightarrow} & -\log(Z_{min}y) = I_{exp}(Z_{min}y) \end{aligned}$$

By item 3, we have that the dual value functions converge:

$$\begin{aligned} & \phi_{2m}(Z_{2m}, \mathcal{Y}_{2m}(x)) \\ = \ & E(-\mathcal{Y}_{2m}(x)Z_{2m}2m(1 - (Z_{2m}\mathcal{Y}_{2m}(x))^{\frac{1}{2m-1}}) - \mathcal{Y}_{2m}(x)^{\frac{2m}{2m-1}} \\ \to \ & E(-\mathcal{Y}_{exp}(x)Z_{min}log(Z_{min}\mathcal{Y}_{2m}(x))) - \mathcal{Y}_{exp}(x) \\ = \ & \phi_{exp}(Z_{min}, \mathcal{Y}_{exp}(x)) \end{aligned}$$

By duality on the 2m-th levels and in the limit, we have convergence of the primal value functionals:

$$\begin{aligned} \lim_{m\to\infty} V_{2m}(x) &= \lim_{m\to\infty} \phi_{2m}(Z_{2m}, \mathcal{Y}_{2m}(x)) \\ &= \phi_{exp}(Z_{min}, \mathcal{Y}_{exp}(x)) = V_{exp}(x) \end{aligned}$$

Further, after we have established the above four steps we have that:

$$\begin{aligned}\lim_{m\to\infty}\mathcal{X}_{2m}(y) &= \lim_{m\to\infty}E(Z_{2m}2m(1-(Z_{2m}y)^{\frac{1}{2m-1}})\\ &= E(\lim_{m\to\infty}Z_{2m}2m(1-(Z_{2m}y)^{\frac{1}{2m-1}})\\ &= E(Z_{min}(-\log(Z_{min}y))) = \mathcal{X}_{exp}(y)\end{aligned}$$

and hence

$$\begin{aligned}X_0^{(2m)}(x) &= I_{2m}(Z_{2m}\mathcal{Y}_{2m}(x))\\ &\overset{a.s./L^1}{\longrightarrow} I_{exp}(Z_{min}\mathcal{Y}_{exp}(x)) = X_0^{exp}(x).\end{aligned}$$

Note, in the deterministic case all kinds of convergence are almost surely. In the non-deterministic case, we know that the dual solutions converge in L^1. We can transfer that to the primal problem and get convergence in L^1 in the primal problem. For Item 3 and 4 it is sufficient to show convergence in probability to establish L^1-convergence, since $(y_m Z_{2m}2m(1-(Z_{2m}y)^{\frac{1}{2m-1}})_m$ and $(2m(1-(Z_{2m}y)^{\frac{1}{2m-1}})_m$, are uniformly integrable:

We start with the case when the terminal value of the trade-off process K_T is deterministic:

Deterministic case:

We proceed with the case, when K_T is deterministic. No further conditions are necessary to prove item 1 to 4 and almost sure convergence of the primal problem:

Later, if $\hat{K}_T$ is not deterministic. We have to impose an assumption - Reverse-Hölder-condition - to the set of martingale measures. In the deterministic case, we can refrain from any integrability assumption, since the Reverse-Hölder-inequality for the minimal martingale measure is satisfied automatically. The $q-$optimal measure coincides with the minimal martingale measure, hence the Reverse-Hölder condition also holds for all $q-$optimal measure with a constant independent of q for $q \in [1, 2]$, see Remark 5.2.2.

We consider the first item. We have to find D as in equation (5.10):

Proof. (item 1.): We consider the function

$$x \cdot 2m(1 - x^{\frac{1}{2m-1}}).$$

We obtain the following estimation:

$$\begin{aligned}
& |x \cdot 2m(1 - x^{\frac{1}{2m-1}})| && (5.11)\\
= & |x \cdot (2m - 2mx^{\frac{1}{2m-1}})| \\
= & x|\frac{2m}{2m-1}\int_x^1 u^{\frac{1}{2m-1}-1}du \cdot 1_{(x\in(0,1))} - \frac{2m}{2m-1}\int_1^x u^{\frac{1}{2m-1}-1}du \cdot 1_{(x\geq 1)}| \\
\leq & x(2\int_x^1 u^{\frac{1}{2m-1}-1}du \cdot 1_{(x\in(0,1))} + 2\int_1^x u^{\frac{1}{2m-1}-1}du \cdot 1_{(x\geq 1)}) \\
\leq & x \cdot x^{-1} \cdot 2(1-x)x^{\frac{1}{2m-1}}1_{(x\in(0,1))} + x \cdot 2(x-1)1_{(x\geq 1)} \\
\leq & 2 \cdot 1 + 2x^2 && (5.12)
\end{aligned}$$

Z_{min} is equal to the minimal martingale measure ($Z_{mmm} = \mathcal{E}(-\int_0^T \hat{\lambda}'_s dM_s)$), which is square-integrable, since it coincides with all q-optimal measures or even more obvious by (4.23) we have:

$$E(|Z_T^{(mmm)}|^{1+\tilde{\mu}}) \leq \exp(\langle \int \hat{\lambda}' dM\rangle_T) = C \in (0,\infty), \quad \text{for all } \tilde{\mu} \in [0,1]$$

We get:

$$\begin{aligned}
& |Z_{2m} 2m(1 - (Z_{2m}y_m)^{\frac{1}{2m-1}}| \\
= & |\frac{1}{y_m} Z_{min} y_m 2m(1 - (Z_{min}y_m)^{\frac{1}{2m-1}}| \\
\leq & 2C_1^{-1} + C_2^2 Z_{min}^2 \equiv D \in L^1(P)
\end{aligned}$$

where C_1, C_2 positive constants. C_1 and C_2 exist since (y_m) is real-valued sequence with a real-valued limit. Later, we do not have that $Z_{2m} = Z_{min}$. We can generalize the above result. For every $\epsilon > 0$, there exists an m_0, choose $m_0 = \frac{1}{2\epsilon} + \frac{1}{2}$, then for all $m \geq m_0$, we have:

$$\begin{aligned}
& |x \cdot 2m(1 - x^{\frac{1}{2m-1}})| \\
\leq & x(2\int_x^1 u^{\frac{1}{2m-1}-1}du \cdot 1_{(x\in(0,1))} + 2\int_1^x u^{\frac{1}{2m-1}-1}du \cdot 1_{(x\geq 1)}) \\
\leq & 2 \cdot x\int_x^1 1^{\frac{1}{2m-1}}u^{-1}du \cdot 1_{(x\in(0,1))} + 2 \cdot x\int_1^x u^{\frac{1}{2m_0-1}-1}du \cdot 1_{(x\geq 1)} \\
\leq & 2 \cdot 1 \cdot x(-\log x)1_{(x\in(0,1))} + x(2(2m_0-1)(x^{\frac{1}{2m_0-1}} - 1))1_{(x\geq 1)} \\
\leq & 2 \cdot 0.4 + 2\epsilon^{-1}x^{\epsilon}x && (5.13)
\end{aligned}$$

Note, $\epsilon = \frac{1}{2m_0-1}$ and $x(-\log x)1_{(x\in(0,1))} \leq 0.4$.

In the deterministic case, this implies both uniformly integrability and the existence of a dominated random variable. Since every constant sequence of a non-negative integrable random variable (in this case $(Z_{min}^{1+\epsilon})_m = (Z_{mmm}^{1+\epsilon})_m$) is uniformly integrable. □

Note, we do not need to prove item 1b, since we already have almost sure convergence. Next, we prove item 2 to 4:

Proof. (item 2. to 4.): Recall from section 4.2, the q-minimal measures coincide with the minimal entropy measure. The sequence $(Z_{2m})_m$ is constant with limit Z_{min}, see Theorem 4.1.18 and notice that the minimal martingale measure and the minimal entropy measure coincide as well, item 2 follows. Next, we treat the third item, for $x, y > 0$:

$$x \quad = \quad \arg \max_{z\in[x,y]} (|(zI_{2m}(z))'|) = \arg \max_{z\in[x,y]} (|zI'_{2m}(z) + I_{2m}(z)|)$$

since

$$\begin{aligned}(zI'_{2m}(z) + I_{2m}(z))' &= \left(z\frac{-2m}{2m-1}z^{\frac{1}{2m-1}-1} + 2m(1 - z^{\frac{1}{2m-1}})\right)' \\ &= \frac{-2m}{(2m-1)^2}z^{\frac{1}{2m-1}-1} - \frac{2m}{2m-1}z^{\frac{1}{2m-1}-1} < 0, \ z > 0.\end{aligned}$$

By an application of the mean value theorem, we have for $x < y$ and $m \geq m_0 = \frac{1}{2\epsilon} + \frac{1}{2}$:

$$\begin{aligned}|xI_{2m}(x) - yI_{2m}(y)| &= |x2m(1 - x^{\frac{1}{2m-1}}) - y2m(1 - y^{\frac{1}{2m-1}})| \\ &\leq |xI'_{2m}(x) + I_{2m}(x)||x - y| \qquad (5.14)\\ &= |x\frac{-2m}{2m-1}x^{\frac{1}{2m-1}-1} + 2m(1 - x^{\frac{1}{2m-1}})||x - y| \\ &\leq (2 \cdot \max\{1, |x^{\frac{1}{2m-1}}|\} + 2\epsilon^{-1}|x^\epsilon| + \frac{0.8}{x})|x - y|\end{aligned}$$

See (5.13) for the second last inequality.

$$\begin{aligned}&|Z_{min}y_m 2m(1 - (Z_{min}y_m)^{\frac{1}{2m-1}}) - (-Z_{min}y\log(Z_{min}y))| \\ \leq\ &|y_m Z_{min}2m(1 - (Z_{min}y_m)^{\frac{1}{2m-1}}) - 2myZ_{min}(1 - (Z_{min}y)^{\frac{1}{2m-1}})| \\ &+|2myZ_{min}(1 - (Z_{min}y)^{\frac{1}{2m-1}}) - (-yZ_{min}\log(Z_{min}y))|)\end{aligned}$$

The second part of the summation converges to zero since $2my(1 - y^{\frac{1}{2m-1}})$ converges to $(-y\log y)$. The first converges by the following consideration (see 5.14):

$$\begin{aligned}&|y_m Z_{min} \cdot 2m(1 - (y_m Z_{min})^{\frac{1}{2m-1}}) - yZ_{min} \cdot 2m(1 - (yZ_{min})^{\frac{1}{2m-1}})| \\ \leq\ &|Z_{min}(y_m - y)|\, 2\Big(\max\{1, (y_m Z_{min})^{\frac{1}{2m-1}}, (yZ_{min})^{\frac{1}{2m-1}}\} \\ &+\epsilon^{-1}(Z_{min}\max\{y, y_m\})^\epsilon + \frac{1}{Z_{min}\min\{y, y_m\}}\Big) \\ \xrightarrow{a.s.}\ &0\end{aligned}$$

since $(y_m)_m$ and y are positive and real-valued.

The convergence is almost sure convergence, since $Z_{2m} = Z_{min}$ almost surely and $(y_m)_m$ is a sequence of real numbers. The fourth item follows since $\frac{a_n}{b_n} \stackrel{a.s.}{\to} \frac{a}{b}$, if $a_n \stackrel{a.s.}{\to} a$, $b_n \stackrel{a.s.}{\to} b$ and $b_n, b > 0$. □

The generalization (5.13) of equation (5.12) is essential when we switch to the non-deterministic case:

Non-deterministic case:

In section 4.2, item 2 is established. We have further seen that usually we do not have that $Z_{2m} = Z_{min} = Z_{mmm}$. In the case with non-deterministic coefficients, we cannot use the nice integrability properties of Z_{mmm}. The refinements of (5.13) are necessary to establish item 1. Also 3 is not obvious: *We show item 1 under the condition that there exists an $\epsilon > 0$ and an $m_1 > 0$ such that Z_{2m} is dominated by a random variable D_1 in $L^{1+\epsilon}(P)$ for all m bigger than m_1:*

Proof. (item 1a assumption: Existence of a dominated random variable): By (5.13), we have for all $m \geq \max(m_0, m_1)$:

$$\begin{aligned} & |Z_{2m} 2m(1-(Z_{2m}y)^{\frac{1}{2m-1}}| \\ \leq\ & 2y^{\epsilon}\epsilon^{-1} Z_{2m}^{1+\epsilon} + 2 \\ \leq\ & 2y^{\epsilon}\epsilon^{-1} D_1 + 2 \equiv D_2 \in L^1(P) \end{aligned} \tag{5.15}$$

□

We prove item 1b and item 2 under the assumption that either $(\log Z_{2m})_m$ and $(Z^{\epsilon}_{2m})_{m \geq m_0}$ are dominated by two integrable random variables or assumption 1.2.4.A and 1.2.4.B resp. 4.2.1:

Proof. (item 1b): We have since $y_m Z_{2m} > 0$ and by the second last inequality of (5.13):

$$\begin{aligned} |2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}| &\leq -2\log(y_m Z_{2m}) 1_{y_m Z_{2m} \in (0,1)} + 2y_m^{\epsilon}\epsilon^{-1} Z_{2m}^{\epsilon} \\ &\leq -2\log(y_m Z_{2m}) 1_{y_m Z_{2m} \in (0,1)} \\ &\quad +2y_m^{\epsilon}\epsilon^{-1}(Z_{2m} 1_{(Z_{2m} \geq 1)} + 1_{(Z_{2m} \in (0,1))}) \\ &\leq -2\log(y_m Z_{2m}) 1_{y_m Z_{2m} \in (0,1)} + 2y_m^{\epsilon}\epsilon^{-1}(Z_{2m}+1) \end{aligned} \tag{5.16}$$

The assertion is trivial under the first assumption. Under the alternative assumption Z_{2m} is uniformly integrable and therefore also $2y_m^{\epsilon}\epsilon^{-1}(1 + Z_{2m})$ (y_m converges to a real number). It remains to show that $-2\log(y_m Z_{2m}) 1_{y_m Z_{2m} \in (0,1)}$ is uniformly integrable. This can be shown in two different ways. The second one even shows that $Z^{(q)} \stackrel{\mathcal{H}^1}{\to} Z^{min}, q \downarrow 1$:

1. We know that $Z_{2m} \xrightarrow{L^1} Z_{min}$ from Theorem 4.2.1. This implies convergence in probability of $2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}$ to $2\log(yZ_{min})1_{yZ_{min}\in(0,1)}$. Further, we know that $-2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}$ is non-negative for all m. It remains to show that

$$E(-2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}) \to E(-2\log(yZ_{min})1_{yZ_{min}\in(0,1)}). \quad (5.17)$$

Then by Theorem 5.2.2, the sequence $-2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}$ is uniformly integrable and therefore also $2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}$. To show (5.17) it suffices to show that $E(\log Z_{2m})$ converges to $E(\log Z_{min})$, since this is satisfied if and only if $E(\log(y_m Z_{2m}))$ converges to $E(\log(yZ_{min}))$ for every real positive sequence $(y_m)_m$ converging to y. Further, $(\log x_n)^- = (\log x_n)1_{x_n\in(0,1)}$ converges if and only if $(\log(x_n)1_{x_n\in[1,\infty)}) = (\log x_n)^+$ and $\log(x_n)$ converge. But for large enough m:

$$\log(y_m Z_{2m})1_{y_m Z_{2m}\in[1,\infty)} \leq (y+C)Z_{2m},$$

$(y+C)Z_{2m}$ is uniformly integrable and we already have convergence in probability, hence the expectation of the positive part converges. So it remains to show that:

$$E(\log Z_{2m}) \to E(\log Z_{min})$$

We have by (4.26) that for $q = \frac{2m}{2m-1}$:

$$\begin{aligned} E(\log Z_{2m}) &= E(M_T^{Q_{\frac{2m}{2m-1}}} - \frac{1}{2}\langle M^{Q_{\frac{2m}{2m-1}}}\rangle_T) \\ &= E(-\int_0^T \hat{\lambda}' dM_s - \frac{1}{q-1}\int_0^T \frac{1}{V_s(q)} d\tilde{m}_s(q) - \frac{1}{2}\langle M^{Q_{\frac{2m}{2m-1}}}\rangle_T) \\ &= E(-\int_0^T \hat{\lambda}' dM_s) - E(\frac{1}{2}\langle M^{Q_{\frac{2m}{2m-1}}}\rangle_T) \end{aligned}$$

since V_t is bounded and therefore the local martingale part $m_t(q)$ is even a martingale with expectation zero. Further, we have by (4.13):

$$\begin{aligned} E(\log Z_{min}) &= E(M_T^{Q_{min}} - \frac{1}{2}\langle M^{Q_{min}}\rangle_T) \\ &= E(-\int_0^T \hat{\lambda}' dM_s - \tilde{m}_T - \frac{1}{2}\langle M^{Q_{min}}\rangle_T) \\ &= E(-\int_0^T \hat{\lambda}' dM_s) - E(\frac{1}{2}\langle M^{Q_{min}}\rangle_T) \end{aligned}$$

So it remains to show that:

$$E(\langle M^{Q_{\frac{2m}{2m-1}}}\rangle_T) \to E(\langle M^{Q_{min}}\rangle_T)$$

Furthermore,

$$\begin{aligned} E(\langle M^{Q_{\frac{2m}{2m-1}}}\rangle_T) &= E(\hat{K}_T - 2\langle -\int \hat{\lambda}' dM_s, \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q)\rangle_T \\ &\quad + \langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q)\rangle_T) \\ &= E(\hat{K}_T + \langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q)\rangle_T) \end{aligned}$$

since $\tilde{m}(q)$ and M are orthogonal and similar:

$$\begin{aligned} E(\langle M^{Q_{min}}\rangle_T) &= E(\hat{K}_T - 2\langle -\int \hat{\lambda}' dM_s, \tilde{m}_s\rangle_T + \langle \tilde{m}\rangle_T) \\ &= E(\hat{K}_T + \langle \tilde{m}_s\rangle_T) \end{aligned}$$

So it remains to show that

$$E(\langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q)\rangle_T - \langle \tilde{m}_s\rangle_T) \to 0,\ q \downarrow 1$$

But,

$$\begin{aligned} &|E(\langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q)\rangle_T - \langle \tilde{m}_s\rangle_T)| \\ \leq\ & E(\langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) - \tilde{m}_s\rangle_T) \to 0, q \downarrow 1 \end{aligned}$$

the last assertion follows by Corollary 2 in Mania et al (2005), which is equivalent to the assertion that M^{Q_q} converges in $\mathcal{H}^2$ to $M^{Q_{min}}$. So the last inequality is just the reverse triangle inequality in $\mathcal{H}^2$, ignoring $\int \hat{\lambda}' dM$.

2. We present a second way to show that $-2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}$ is uniformly integrable, which in addition shows the convergence of $Z^{(q)}$ to $Z^{(min)}$ in $\mathcal{H}^1$. We show that:

$$\log Z_{2m} \stackrel{\mathcal{H}^1}{\to} \log Z_{min} \tag{5.18}$$

(without using that $Z_{2m} \stackrel{\mathcal{H}^1}{\to} Z_{min}$.) This yields convergence in probability of $-2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}$ to $-2\log(yZ_{min})1_{yZ_{min}\in(0,1)}$ and L^1-integrability. Further, we know that $-2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}$ is non-negative for all m. As before by Theorem 5.2.2 it remains to show that

$$E(-2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}) \to E(-2\log(yZ_{min})1_{yZ_{min}\in(0,1)}) \tag{5.19}$$

and again as before this holds if

$$E(\log Z_{2m}) \to E(\log Z_{min})$$

which is satisfied by (5.18).

To show (5.18), we have by (4.26) that for $q = \frac{2m}{2m-1}$:

$$\begin{aligned}\log Z_{2m} &= M^{Q_{\frac{2m}{2m-1}}} - \frac{1}{2}\langle M^{Q_{\frac{2m}{2m-1}}}\rangle_T \\ &= -\int_0^T \hat{\lambda}' dM_s - \frac{1}{q-1}\int_0^T \frac{1}{V_s(q)} d\tilde{m}_s(q) - \frac{1}{2}\langle M^{Q_{\frac{2m}{2m-1}}}\rangle_T\end{aligned}$$

Further, we have by (4.13):

$$\begin{aligned}\log Z_{min} &= M^{Q_{min}} - \frac{1}{2}\langle M^{Q_{min}}\rangle_T \\ &= -\int_0^T \hat{\lambda}' dM_s - \tilde{m}_T - \frac{1}{2}\langle M^{Q_{min}}\rangle_T\end{aligned}$$

Furthermore,

$$\begin{aligned}\langle M^{Q_{\frac{2m}{2m-1}}}\rangle_T &= \hat{K}_T - 2\langle -\int \hat{\lambda}' dM_s, \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q)\rangle_T \\ &\quad + \langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q)\rangle_T \\ &= \hat{K}_T + \langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q)\rangle_T\end{aligned}$$

since $\tilde{m}(q)$ and M are orthogonal and similar:

$$\begin{aligned}\langle M^{Q_{min}}\rangle_T &= \hat{K}_T - 2\langle -\int \hat{\lambda}' dM_s, \tilde{m}_s\rangle_T + \langle \tilde{m}\rangle_T \\ &= \hat{K}_T + \langle \tilde{m}_s\rangle_T\end{aligned}$$

Finally, let $Z^{(2m)} = (E(Z_{2m}|\mathcal{F}_t))_t$ and $Z^{min} = (E(Z_{min}|\mathcal{F}_t))_t$:

$$\begin{aligned}&\|\log Z^{(2m)} - \log Z^{min}\|_{\mathcal{S}^1} \\ \le\ &\|\log Z^{(2m)} - \log Z^{min}\|_{\mathcal{H}^1} \\ =\ &\|M^{Q_q} - \frac{1}{2}\langle M^{Q_q}\rangle - M^{Q_{min}} + \frac{1}{2}\langle M^{Q_{min}}\rangle\|_{\mathcal{H}^1} \\ =\ &\| - \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) + \tilde{m} + \frac{1}{2}(\langle M^{Q_{min}}\rangle - \langle M^{Q_q}\rangle)\|_{\mathcal{H}^1} \\ \le\ &\| - \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) + \tilde{m}\|_{\mathcal{H}^1} + \|\frac{1}{2}(\langle M^{Q_{min}}\rangle - \langle M^{Q_q}\rangle)\|_{\mathcal{H}^1} \\ \le\ &\| - \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) + \tilde{m}\|_{\mathcal{H}^2} + \frac{1}{2}\|(\langle M^{Q_{min}} >_T - \langle M^{Q_q}\rangle_T)\|_{L^1} \\ =\ &E(\langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) - \tilde{m}\rangle_T) \\ &+ \frac{1}{2}E(|\langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q)\rangle_T - \langle \tilde{m}\rangle_T|) \\ \le\ &\frac{3}{2}E(\langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) - \tilde{m}\rangle_T) \to 0,\ q \downarrow 1\end{aligned}$$

by Corollary 2 in Mania et al (2005). It follows that $Z_{2m} \xrightarrow{P} Z_{min}$ and since $(Z_{2m})_m$ is uniformly integrable, by Theorem 5.2.2 we have $Z_{2m} \xrightarrow{L^1} Z_{min}$. Using that $\sup_t Z_t^{(2m)}$ is uniformly integrable, Doob's inequality yields convergence in $\mathcal{H}^1$ of $Z^{(q)}$ to Z^{min}. That is valid for every $q_m \downarrow 1$!

□

We went away from the deterministic case: The sequence $(Z_{2m})_m$ is not constant in general. The assumption used to show item 1a anticipates integrability conditions upon $\frac{2m}{2m-1}-$minimal measure for all m bigger than a certain m_0. The definition of the problem only gives us that $Z_{2m} \in L^q(P)$, $q = \frac{2m}{2m-1}$. However, the assumption requires that $Z_{2m} \in L^{q_0}(P)$, $q_0 = \frac{2m_0}{2m_0-1}$. This is a rather strong assumption, we do not know when it is satisfied, except in the deterministic case. To get away from this assumption, it might be successful to have a closer look at $q-$minimal measure, e.g. in Mania et al (2003a) or Grandits and Rheinländer (2002a). However, it is more elegant to prove uniform integrability instead. To do this, we need some preparation and have to impose an additional assumption (assumption 1.2.4.B or 4.2.1). We cite a theorem from Shiryaev (1996):

Theorem 5.2.3. *Let $\xi_1, \xi_2, ...$ be a sequence of integrable random variables and $G = G(t)$ a nonnegative increasing function, defined for $t \geq 0$, such that:*

$$\lim_{t\to\infty} \frac{G(t)}{t} = \infty, \ \sup_n E(G(|\xi_n|)) < \infty$$

Then the family $(\xi_n)_n$ is uniformly integrable.

It is sufficient to show that $Z_{2m}^{1+\epsilon}$ satisfies these conditions, because $Z_{2m} 2m(1 - (Z_{2m}y)^{\frac{1}{2m-1}}$ is dominated by $2y^\epsilon \epsilon^{-1} Z_{2m}^{1+\epsilon}$ (see (5.15)) and $Z_{2m} 2m(1-(Z_{2m}y)^{\frac{1}{2m-1}}$ is therefore also uniformly integrable. We use Theorem 5.2.3. To establish all properties of this theorem assumption 4.2.1 is sufficient: All P-martingales are continuous (e.g. if the underlying filtration is continuous) and there exists a $Q \in \mathcal{M}^e_{q_0}(P)$ for a $q_0 > 1$ and its density process Z satisfies the Log-Reverse Hölder condition $(R_{LLogL}(P))$ with constant K, i.e. there exists a $K > 0$ such that

$$\sup_{\tau\in\mathcal{T}} E_P(\frac{Z_T}{Z_\tau} \log^+ \frac{Z_T}{Z_\tau} | F_\tau) < K. \tag{5.20}$$

$\mathcal{T}$ is the set of all stopping times. We further defined the Reverse Hölder inequality $R_p(Q)$: As a reminder, Z satisfies the Reverse Hölder inequality $R_p(Q)$, if there exists a $C > 0$ such that

$$\sup_{\tau\in\mathcal{T}} E_Q(|\frac{Z_T}{Z_\tau}|^p | F_\tau) < C. \tag{5.21}$$

As seen before the Log-Reverse-Hölder inequality implies the Reverse Hölder inequality for some $p_0 > 1$, see Proposition 1 Mania et al (2005) or similar Lemma 2.2. Grandits and Rheinländer (2002a), i.e. assumption 4.2.1 implies assumption 1.2.4.B However, to use the result in Mania et al (2005), we also need that 1.2.4.A is satisfied:

Under assumption 4.2.1 or assumption 1.2.4, item 1a holds:

Proof. (item 1a): Under assumption 4.2.1, Grandits and Rheinländer (2002a) prove that all q-optimal martingale measures with $1 < q \leq q_0$ satisfy the Log-Reverse Hölder inequality with a constant independent of q, see Lemma 4.8.:

$$\sup_{1<q\leq q_0} \sup_{\tau\in\mathcal{T}} E(\frac{Z_T^{(q)}}{Z_\tau^{(q)}} \log^+ \frac{Z_T^{(q)}}{Z_\tau^{(q)}} | F_\tau) < C \tag{5.22}$$

By Lemma 4.2.3 there exists a $\tilde{\mu}$ such that all q-optimal measures Z_q with $q \leq q_0$ satisfy the Reverse Hölder condition $R_{1+\tilde{\mu}}(P)$, i.e.

$$\sup_{1<q\leq q_0} \sup_{\tau\in\mathcal{T}} E(|\frac{Z_T^{(q)}}{Z_\tau^{(q)}}|^{1+\tilde{\mu}} | F_\tau) < K. \tag{5.23}$$

Since the single constants K_q are only dependent of C and independent of q, we set $K \equiv K_q$. Hence,

$$\sup_{1<q\leq q_0} E(|Z_T^{(q)}|^{1+\tilde{\mu}}) < K. \tag{5.24}$$

It turns out that assumption 1.2.4 is sufficient for the last assertion, see Mania et al (2005) proof of Theorem 1, together with Kazamaki (1994).

Next, we apply Theorem 5.2.3 to the following function:

$$G(t) = t^{1+\epsilon_2}$$

where $\epsilon_2 > 0$ still arbitrary. $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$ is trivially satisfied. We liked to prove that $(Z_{(q)}^{1+\mu})_{q\leq q_0}$ is uniformly integrable for a $\mu > 0$. So we have to show that:

$$\sup_{q\leq q_0} E(G(|Z_q|^{1+\mu})) = \sup_{q\leq q_0} E((|Z_q|^{1+\mu})^{1+\epsilon_2}) < \infty.$$

But choose $\epsilon_2 > 0$ and $\mu > 0$ such that

$$\tilde{\mu} = \mu + \epsilon_2 + \mu\epsilon_2 = (1+\mu)(1+\epsilon_2) - 1 \tag{5.25}$$

and the assertion follows from (5.24) and Theorem 5.2.3. □

Remark 5.2.2. In the deterministic case, we do not need to assume the Reverse-Hölder-condition, since it is satisfied automatically, see example 4.1.2. Further, we have 4.23:

$$\begin{aligned}\sup_{1<q\leq q_0} E(|Z_T^{(q)}|^{1+\tilde{\mu}}) &= E(|Z_T^{(mmm)}|^{1+\tilde{\mu}}) \\ &\leq \exp(\langle \int \hat{\lambda}' dM\rangle_T) = C \in (0,\infty), \quad \text{for all } \tilde{\mu} \in [0,1]\end{aligned}$$

item 1a follows as above, without any integrability conditions on the minimal martingale measure.

We switch to item 3:

Under the assumptions of Item 1 and 2. Item 3 is satisfied for convergence in L^1:

Proof. (item 3):

By (5.14), we have:

$$\begin{aligned}
& |Z_{2m}y_m 2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}) - (-Z_{min}y\log(Z_{min}y))| \\
\leq\ & |y_m 2m Z_{2m}(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}) - 2mZ_{min}y(1-(Z_{min}y)^{\frac{1}{2m-1}}) \\
& +2mZ_{min}y(1-(Z_{min}y)^{\frac{1}{2m-1}}) - (-Z_{min}y\log(Z_{min}y))| \\
\leq\ & |y_m Z_{2m}2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}) - 2myZ_{min}(1-(Z_{min}y)^{\frac{1}{2m-1}})| \\
& +|2myZ_{min}(1-(Z_{min}y)^{\frac{1}{2m-1}}) - (-yZ_{min}\log(Z_{min}y))|) \\
\leq\ & 2\Big(\max\{1,(y_mZ_{2m})^{\frac{1}{2m-1}},(yZ_{min})^{\frac{1}{2m-1}}\} \qquad (5.26) \\
& +\epsilon^{-1}(\max\{yZ_{min},y_mZ_{2m}\})^{\epsilon} + \frac{1}{\min\{Z_{min}y, Z_{2m}y_m\}}\Big)|Z_{2m}y_m - Z_{min}y| \\
& +|2myZ_{min}(1-(Z_{min}y)^{\frac{1}{2m-1}}) - (-yZ_{min}\log(Z_{min}y))| \\
\xrightarrow{P}\ & 0 \qquad (5.27)
\end{aligned}$$

for any positive, real-valued sequence $(y_m)_m$ with limit y, e.g. $(\mathcal{Y}_{2m}(x))$ for fixed x. Since $Z_{2m}y_m2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}})$ is uniformly integrable, convergence in L^1 follows. □

We can even get convergence in $L^{1+\mu}$. One can prove by simple integration theory that:

$$\begin{aligned} & |x\cdot 2m(1-x^{\frac{1}{2m-1}}) - y\cdot 2m(1-y^{\frac{1}{2m-1}})| \\ \leq\ & 4\,|(x-y)|\max\{1,x^{\frac{1}{2m-1}},y^{\frac{1}{2m-1}}\} \qquad (5.28)\end{aligned}$$

and Hölder's inequality for $p = 2m-1$ yields $L^{1+\mu}$−convergence.

Next, we prove item 4 under the conditions of item 1 to 3:

$$I_{2m}(Z_{2m}y) = 2m(1-(Z_{2m}y)^{\frac{1}{2m-1}}) \stackrel{L^1/a.s.}{\longrightarrow} -\log(Z_{min}y) = I_{exp}(Z_{min}y).$$

Proof. (item 4.): From item 2, we know that $Z_{2m} \xrightarrow{L^1} Z_{min}$ and under our conditions all Z_{2m} and Z_{min} are densities of equivalent martingale measures and therefore strictly positive. Hence,

$$\begin{aligned} I_{2m}(Z_{2m}y_m) &= 2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}) \\ &= \frac{2my_mZ_{2m}(1-(Z_{2m}y_m)^{\frac{1}{2m-1}})}{y_mZ_{2m}} \\ &\xrightarrow{P} \frac{-yZ_{min}\log(Z_{min}y)}{yZ_{min}} \\ &= -\log(Z_{min}y) = I_{exp}(Z_{min}y) \end{aligned} \tag{5.29}$$

for any positive and real-valued sequence $(y_m)_m$ with limit y. The L^1 convergence follows since $I_{2m}(Z_{2m}y_m)$ is uniformly integrable, see item 1b. □

Note, in the deterministic case - we have $Z_{2m} = Z_{min}$, no further assumptions, like the Log-Reverse-Hölder-inequality, are necessary. In both cases, we get from item 1 to 4:

$$\begin{aligned} \lim_{m\to\infty} \mathcal{X}_{2m}(y) &= \lim_{m\to\infty} E(Z_{2m}2m(1-(Z_{2m}y)^{\frac{1}{2m-1}})) \\ &(= \lim_{m\to\infty} E(Z_{min}2m(1-(Z_{min}y)^{\frac{1}{2m-1}}))) \\ &= E(\lim_{m\to\infty} Z_{2m}2m(1-(Zy)^{\frac{1}{2m-1}})) \\ &= E(Z_{min}(-\log(yZ_{min}))) \\ &= -\log y - \min H(Q|P) = \mathcal{X}_{exp}(y) \end{aligned} \tag{5.30}$$

But under the imposed assumption, by (5.30) it follows that

$$\mathcal{Y}_{2m}(x) \to \mathcal{Y}_{exp}(x)$$

So together with (5.30), we have in the deterministic case:

$$\begin{aligned} X_0^{(2m)}(x) &= I_{2m}(Z_{(2m)}\mathcal{Y}_{(2m)}(x)) \\ &\xrightarrow{a.s.} I_{exp}(Z_{min}\mathcal{Y}_{exp}(x)) = X_0^{exp}(x) \end{aligned}$$

In the non-deterministic-case, we get by item 4:

$$I_{2m}(Z_{2m}\mathcal{Y}_2m(x)) \stackrel{L^1}{\to} I_{exp}(Z_{min}\mathcal{Y}_{exp}(x))$$

Finally,

$$\begin{aligned} X_0^{(2m)}(x) &= I_{2m}(Z_{(2m)}\mathcal{Y}_{(2m)}(x)) \\ &\xrightarrow{L^1} I_{exp}(Z_{min}\mathcal{Y}_{exp}(x)) = X_0^{exp}(x) \end{aligned}$$

We summarize in the following theorem:

Theorem 5.2.4. *In a model with an $\mathcal{F}_t$-adapted continuous semimartingale $S = S_0 + M + A$ as in section 1.2.1 assume that the set of all local equivalent martingale measures is nonempty (assumption 1.2.3). Further, suppose that one of the following assumption holds:*

- *There exists an $\epsilon > 0$ and an m_0 such that for all $m \geq m_0$, the sequences $(Z_{2m}^{\epsilon})_m$ and $(|\log Z_{2m}|)_m$, where Z_{2m} the density of the $\frac{2m}{2m-1}$-minimal martingale measure, are dominated by L^1 random variables.*
- *Assumption 1.2.4.A and 1.2.4.B resp.4.2.1 are satisfied.*
- *The terminal value of the mean variance tradeoff process ($\hat{K}_T = \langle -\int \hat{\lambda}' dM \rangle_T$) is deterministic.*

Then, the solution of the utility maximization problem of the $2m$-approximating sequence of the exponential function converges in L^1 to the solution of the exponential problem, i.e.:

$$X_0^{(2m)}(x) = 2m - 2m \left(Z_{2m} \left(\frac{2m - x}{2mE\left(Z_{2m}^{\frac{2m}{2m-1}}\right)} \right)^{2m-1} \right)^{\frac{1}{2m-1}}$$

$$\xrightarrow{L^1} X_0^{exp}(x) = -\log Z_{min} + H(Q_{min}|P) + x$$

The convergence is almost surely, if the trade-off process K_T is deterministic, e.g. in a Brownian setting with deterministic coefficients: μ, σ, r only depend on t. We have that the dual problem of the approximating sequence and of the exponential utility function has the same solution (only considering the density-part Z, not y), the density of the minimal entropy measure. Consequently, the solution of the primal problem on $2m$-th level converges to primal solution of the exponential problem:

$$X_0^{(2m)}(x) = 2m - 2m \left(Z_{min} \left(\frac{2m - x}{2mE\left(Z_{min}^{\frac{2m}{2m-1}}\right)} \right)^{2m-1} \right)^{\frac{1}{2m-1}}$$

$$\xrightarrow{a.s.} X_0^{exp}(x) = -\log Z_{min} + H(Q_{min}|P) + x.$$

The values of the dual problems converge;

$$\lim_{m\to\infty} \phi_{2m}(y_m, Z_{2m-opt}) = \phi_{exp}(Z_{min}, y),$$

so also the value functions of the primal problem:

$$\lim_{m\to\infty} E(u_{2m}(X_0^{(2m)})(x)) = \lim_{m\to\infty} V_{2m}(x) = V_{exp}(x) = E(U_{exp}(X_0^{exp})(x)).$$

5.2.1 Some Comments on the Convergence of the Hedging Problem

In a forthcoming paper, we will then consider how the hedging problem of the $2m$-th approximating sequence and the hedging problem of the exponential function are connected. We give the idea: In the case of the approximating sequence of the exponential function $-(1-\frac{\alpha x}{2m})^{2m}$, we have:

$$-(1-\frac{\alpha x}{2m})^{2m}(1+\frac{\alpha\xi}{2m})^{2m} = -(1-\frac{\alpha(x-\xi)}{2m}-\frac{\alpha\xi}{4m^2})^{2m} \overset{a.s.}{\to} -e^{-\alpha(x-\xi)}$$

So we can define:

$$Z^{\xi,m} := \frac{dP^{\xi}_{2m}}{dP} = c^{\xi}_{2m}(1+\frac{\alpha\xi}{2m})^{2m},\ c^{\xi}_{2m} = (E((1+\frac{\alpha\xi}{2m})^{2m}))^{-1}$$

Under assumption 3.2.5, the induced densities of the measures converge almost surely to the density of P_ξ as defined in (3.67). Further, set $q=\frac{2m}{2m-1}$, from Theorem 5.2.4 we know that the optimal solution $X_0^{(2m),\xi}(x)$ of

$$\begin{aligned} &\max_{X\in L^{2m}(P_\xi)|\sup_{Q\in\mathcal{M}^q_e} E_Q(X)\le x} E(\frac{e^{\alpha\xi}}{E(e^{\alpha\xi})}(-1)(1-\frac{\alpha X}{2m})^{2m}) \\ = &\max_{X\in L^{2m}(P_\xi)|\sup_{Q\in\mathcal{M}^q_e} E_Q(X)\le x} E_{P^\xi}(-(1-\frac{\alpha X}{2m})^{2m}) \qquad (5.31) \end{aligned}$$

uniquely exists and converges to the solution $(X_\xi(x))$:

$$\begin{aligned} &\max_{X\in L^{p}(P_\xi)|\sup_{Q\in P_{\xi_f}} E_Q(X)\le x} E_{P^\xi}(e^{-\alpha X}) \\ = &\max_{X\in L^{p}(P_\xi)|\sup_{Q\in P_{f}} E_Q(X)\le x} E(e^{-\alpha(X-\xi)}) \end{aligned}$$

for an arbitrary $p\in(1,\infty]$. Question is now if

$$\begin{aligned} &\max_{X^n\in L^{2m}(P^\xi_{2n})|\sup_{Q\in\mathcal{M}^q_e} E_Q(X^n)\le x} E(c^\xi_{2n}(1+\frac{\alpha\xi}{2n})^{2n}(-1)(1-\frac{\alpha X^n}{2m})^{2m}) \\ = &\max_{X^n\in L^{2m}(P^\xi_{2n})|\sup_{Q\in\mathcal{M}^q_e} E_Q(X^n)\le x} E_{P^\xi_{2n}}(-(1-\frac{\alpha X^n}{2m})^{2m}) \end{aligned}$$

converges to (5.31) if n tends to infinity using that $Z^{\xi,n} \overset{a.s.}{\to} \frac{e^{\alpha\xi}}{E(e^{\alpha\xi})}$. In particular, if $\lim X^n =: \tilde{X} \in L^{2m}(P_\xi)$. Then, we can choose a diagonal sequence and are done.

Summary and Conclusion

We give an overview of this thesis, illustrate the results, and draw a conclusion:

Chapter 1

We consider three problems that arise in Economics: A pure investment problem, a pricing problem of a contingent claims ξ, and a mixture of both. To formulate an optimization problem, we need to know which market we face and what kind of attitude towards risk and return the investors have. After introducing the notation to be used (section 1.1), Chapter 1 presents the market model (section 1.2.1) and briefly explains the concept of optimizing expected utility (section 1.3). Possible investments are n stocks, modelled by an $n-$dimensional semimartingale and a constant bond $(S, 1)$ on a probability space $(\Omega, \mathcal{F}, P)$. We define a class of allowable self-financing investment $\mathcal{A}^p$ (and consumption $\mathcal{K}^p$) strategies. Processes $(\mathcal{W}_{(C)}(x))$ of the form $Y = x + \int NdS(-C), (N, C) \in \mathcal{A}^p \times \mathcal{K}^p$ then describe investors' wealth. This is specialized to a Brownian Model in section 1.2.2. We assume that the market is arbitrage-free, but distinguish between complete and incomplete markets, i.e. in a complete market all contingent claims, $\mathcal{F}_T$- measurable, $L^p(P)$-integrable random variables, are attainable by an allowable strategy in $\mathcal{A}^p$ or equivalently there is a unique martingale measure. The following main problems are considered, see (1.36) and (1.37):

1. Pure investment problem
$$V(x) = \sup_{Y \in \mathcal{W}(x)} E(U(Y_T)) \tag{5.32}$$

2. Pricing Problem and Mixed Problem (using different utility functions)
$$V_{\xi,C}(x) = \sup_{Y \in \mathcal{W}_C(x)} E(U(Y_T - \xi)), \tag{5.33}$$

Chapter 2

Chapter 2 describes (5.32) and (5.33) in detail, see problem 2.3, and suggests appropriate utility functions to model them. Problem 2.3 is a dynamic problem, i.e. we optimize over wealth processes. To get rid of the time component we formulate an equivalent optimization problem under constraints over a class of random variables ($\mathcal{G}_C^p(x)$) - a static problem. One direction is trivial. We still have to show that there exists a hedging (superhedging) strategy for every random variable f in $\mathcal{G}_C^p(x)$ (see section 2.1.1): There exists an allowable investment fstrategy (and a consumption process) such that the terminal value of the corresponding wealth process is equal to f. Theorem 2.2.1 treats the complete case. For hedging strategies, Lemma 2.2.3 (Brownian case) and Theorem 2.2.5 (semimartingale model) deal with the result in an incomplete market. Whereas the result for hedging strategies directly follows from a known Theorem (here Theorem 2.2.4), the L^p-superhedging case is firstly stated in this thesis: We imitate a result from El Karoui and Quenez (1995) to find a supermartingale (Theorem 2.2.7), which has the form of a wealth process: $Y = x + \int NdS - C$ (Theorem 2.2.8). We finally show the right integrability of this supermartingale in Theorem 2.2.9 and obtain the main result of this chapter:

$$\begin{array}{ll} \text{Dynamic Problem:} & \sup_{Y\in\mathcal{W}_C(x)} E[U(Y_T-\xi)] \\ & \Big\Updownarrow \ \text{Theorem 2.2.10} \\ \text{Static Problem:} & \sup_{f\in\mathcal{G}_C^p(x)} E[U(f-\xi)] \end{array} \tag{5.34}$$

where

- $\diamond$ $\mathcal{W}_C(x) = \{Y | Y_t = x + \int_0^t NdS - C_t,\ N \in \mathcal{A}^p,\ C \in \mathcal{K}^p\}$
- $\diamond$ $\mathcal{G}_C^p(x) = \{f \in L_p(\mathcal{F}_T), \forall Q \in \mathcal{M}_e^q\ E_Q(f) \le x\}$, $x \in \mathbb{R}$ ($\mathbb{R}^+$).

Chapter 3

In this chapter, we suggest a new way to solve the static problem using methods from convex analysis. We have chosen the market model such that the static problem (5.34) is a convex optimization problem under constraints over the reflexive Banach space L^p. After putting the problem into the framework of convex analysis, we set up the saddlepoint problem of the corresponding Lagrange-functional. To find a saddlepoint, we finally have to solve a dual problem ($\min_{\lambda\in\mathbb{R}^+\mathcal{M}_{a,Z}^q} \phi(\lambda)$):

$$\begin{array}{ccccc} \text{static problem} & \xrightarrow{\text{Th. 3.1.4}} & \text{saddlepoint problem} & \xrightarrow{(3.20)} & \text{dual problem} \\ \Big\uparrow \text{optimal solution} & & \text{saddle}\Big\uparrow\text{point} & & {\scriptstyle (3.21)}\Big\downarrow \\ X_0(x) := X_0(\lambda_0^*) & \xleftarrow{(3.22)} & (\lambda_0^*, X_0(\lambda_0^*)) & \xleftarrow{(3.16)} & \lambda_0^* = \mathcal{Y}(x) Z_{\mathcal{Y}(x)} \end{array}$$

Note, $\mathbb{R}^+\mathcal{M}_{a,Z}^q \subset L^q$. The above problem is solved for $\xi \equiv 0$. Section 3.2.3 therefore presents some ideas to treat the case $\xi \neq 0$. The method is applied to some examples, e.g. the exponential utility function.

Chapter 4

This chapter introduces three candidates for a solution of the dual problem: The densities of the minimal martingale measure Z_T^{mmm}, the minimal entropy measure $Z_{min} := Z_T^{min}$, and the q-optimal measure $Z_q := Z_T^{(q)}$ for different $q > 1$.

Let

$$\hat{Z}_t = \exp\{-\int_0^t \hat{\lambda}_s' dM_s - \frac{1}{2}\int_0^t \|\hat{\lambda}\|^2 d\langle M\rangle_s\},$$

be a martingale, where $\hat{\lambda}$ is defined in (2.17), then the *minimal martingale measure* is defined as

$$dQ_{mmm} := d\hat{Q} = \hat{Z}_T dP \quad \text{(Definition 4.1.5)}.$$

This generalizes the intuitive meaning introduced in Föllmer and Schweizer (1991) that Q_{mmm} preserves the structure of P best (certain $P-$martingales stay Q_{mmm}-martingales), see definition 4.1.2 (one-dimensional case) and 4.1.6 (multidimensional case) or definition 4.1.3 and 4.1.7 in the Brownian case, respectively. In the Brownian case, some further characterizations are given (see Definition 4.1.4 and 4.1.8).

The *minimal entropy measure* Q is the unique measure, minimizing the relative entropy H with respect to P (see Definition 4.1.9):

$$Q_{min} = \arg \min_{Q \in P_f(P)} H(Q|P)$$

(provided there exists a martingale measure with finite entropy). Relative entropy is used in information theory. So in some sense Q_{min} is the martingale measure carrying most information about P. The minimal entropy measure can be characterized by a backward stochastic differential equation (BSDE), see Theorem 4.1.10.

Finally, we discuss the existence of the $q-$*optimal measure* (see Theorem 4.1.12 and 4.1.13):

$$Q^q = \arg \min_{Q \in \mathcal{M}_e^q} E((\frac{dQ}{dP})^q)$$

To ensure that the q-optimal measure Q^q, with density $Z_T^{(q)}$ is in fact an (equivalent) measure ($Z_T^{(q)} > 0$), we need to impose the Reverse-Hölder-condition (assumption 1.2.4.B). A characterization via a BSDE-approach is also possible, see Theorem 4.1.14.

Relations: If $\langle -\int \hat{\lambda}_s' dM_s\rangle_T$ is deterministic, then all suggested measures coincide: Q_{mmm}, Q_{min}, and all Q_q for $q > 1$, see (4.14), Theorem 4.1.11, and Theorem 4.1.19. In the nondeterministic case, section 4.2 shows that under assumption 1.2.4 (Theorem 4.2.1):

$$Z_T^{(q)} \stackrel{L^1}{\rightarrow} Z_T^{min},\ q \downarrow 1 \tag{5.35}$$

Chapter 5

The content of the fifth Chapter is, to our best knowledge, entirely new. We know that the sequence $(-(1-\frac{x}{2m})^{2m})_m$ converges to $-e^{-x}$. The aim of this Chapter is to prove that also the solutions of the corresponding utility maximization problem converge:

1. 2mth problem:

$$V_{2m}(x) = \max_{X \in L^{2m}(P): \forall Z \in \mathcal{M}_{a,Z}^{\frac{2m}{2m-1}} : E(ZX) \leq x} E\left(-\left(1-\frac{X}{2m}\right)^{2m}\right)$$

2. Exponential problem:

$$V_{exp}(x) = \max_{X \in L^p(P): \forall Z \in \mathcal{M}_{a,Z}^{q} : E(ZX) \leq x} E(-e^{-X}),\ p \in [1,\infty)$$

In section 5.1, we derive the solution of the 2m-th problem (the solution of the exponential problem was derived in Chapter 3). The dual solution is the $\frac{2m}{2m-1}$-optimal martingale measure. The minimal entropy measure is the dual solution to the exponential problem. So (5.35) shows the L^1-convergence of the solutions of the dual problem! In section 5.2 we prove that L^1-convergence can be transferred to the primal problem. We can even show almost sure convergence, if $\langle -\int_0^T \hat{\lambda}'_s dM_s \rangle$ is deterministic. In this case, Chapter 4 shows that the minimal martingale measure, the minimal entropy measure, and all q-optimal measures for different $q > 1$ coincide. The dual solutions are equal almost surely. This leads to almost sure convergence in the primal problem. $\langle -\int_0^T \hat{\lambda}'_s dM_s \rangle$ is deterministic, e.g in the Brownian case with deterministic drift and volatility. The value function of the primal problem converges, because of the duality on all levels and in the limit and the convergence on the dual side.

Theorem 5.2.4: Under assumption 1.2.4, we have

$$\begin{array}{ccc} X_0^{(2m)}(x) & \longrightarrow & X_0^{(exp)}(x) \\ \cong & \circlearrowleft & \cong \\ Z_{2m} & \longrightarrow & Z_{min} \end{array} \tag{5.36}$$

and

$$\begin{array}{ccc} V_{2m}(x) & \longrightarrow & V_{exp}(x) \\ = & & = \\ \phi_{2m}(\mathcal{Y}_{2m}(x)) & \longrightarrow & \phi_{exp}(\mathcal{Y}_{exp}(x)) \end{array} \tag{5.37}$$

Finally, we give an outlook for the convergence of the hedging problem.

Conclusion

Chapter 1 to 4 form the framework for Theorem 5.2.4. Chapter 2 proves the reformulation of the dynamic problem to the static problem. In our particular setting (introduced in chapter 1), this result is new. Chapter 3 provides the reader with a new way of solving the static problem. In particular, chapter 1-4 explain how to solve the exponential problem and the problem of its approximating sequence as a slight modification of the isoelastic utility functions. Whereas the basic idea of chapter 1-4 is well-known, the connection between the approximating sequence and the exponential utility function, described in chapter 5, is mainly undiscovered. A portfolio for iso-elastic utility functions was derived in Bürkel (2004), one can also derive a portfolio using the value function. An explicit portfolio for the exponential case N^*_{exp} is still an open problem. So far this problem was only solved in some very special cases, see e.g. Delbaen et al (2002) p.118/119 and Rouge and El Karoui (2000). Investment and Hedging Problem is very well-described by an exponential utility function. So on the long-term basis the goal is to derive a portfolio for the exponential case in a more general setting using an approximation of the portfolio N^*_{2m} for the approximating sequence, where N^*_{2m} is obtained, by adjusting the already established portfolio for the isoelastic case. In a Brownian setting with constant coefficient, only one stock and one Brownian motion (complete market) one can easily calculate all portfolios explicitly and a convergence result holds true:

$$N^*_{2m} = \frac{(\mu - r)2m}{\sigma^2(2m-1)} \to \frac{\mu - r}{\sigma^2} = N^*_{exp}$$

Diagram 5.36 and 5.37 present results on the relation between various kinds of optimal martingale measures and the solution of a modification of the isoelastic and the exponential problem. The diagram already shows the convergence of the terminal values and the value functions. Next, one could modify the results in Bürkel (2004) on the isoelastic case and use the convergence result on the terminal values to derive an explicit portfolio in the exponential case. Furthermore, one can now try to derive a portfolio for the 2mth problem from the value function and use the relationship of the value functions to establish an explicit portfolio for exponential utility functions.

The complete description of the interrelation given in Theorem 5.2.4 is first stated in this thesis. The present work, so, may be also seen as a contribution towards a solution of the open problem of finding an explicit portfolio for the exponential utility function.

Appendix A

Some Economic Aspects

In the appendix, we explain why we use utility functions to model our preferences. We answer the following question which arose in the beginning. How shall we model that investor 1 prefers investment A against investment B and a second investor 2 does it the other way around? One answer is that investment A yields a high expected return, but also keeps a lot of risk. Investment B contains less risk, say no risk, but also less return. So investor 1 is willing to bear more risk for a higher return. Investor 2 is more dependent on a safe return, so he prefers investment B. However, he might switch to investment A, if the costs to have a safe investment are too high. Consider the following example: You toss a dice, your gain is the appearing number. So if you play this game very often your expected gain is 3.5. Now if we offer you 3.5 that you do not play this game, you might say no because you like gambling. To exclude that say your gain is 3.5 million. We allege you would sell in this case, we say you are risk-averse. If you are indifferent, we say you are risk-neutral and last but not least you are risk-seeking if you even want that we pay you more than the expected return to switch to the safe investment. Usually it is assumed that individuals are risk-averse.

Back to our example, what happens if we only offer you 3 million, since we have to bear your risk or even only 2 million? How would you decide? This depends on your preference structure. We have seen that preference structures have certain properties, provided rational behavior of the investors is assumed. So first we introduce usual properties of a preference structure. Whereas we should never forget that individuals do not behave rational in a lot of cases (St-Petersburg-Paradox see below), however without this assumption a consideration of preferences is not possible. We assume rational behavior and introduce the concept of a utility function and the idea of maximizing expected utility. Except maximization of our wealth at a certain time T (pure utility maximization problem), we could also imagine that we want to buy a product and find out an acceptable price for it (pricing problem via an approximate hedge). Finally, the problem of maximizing our wealth under the constraint the we also have to buy products or face claims (e.g. telephone bill) respectively (mixed problem). These the three problems are explained from a more economic point of view in section A.2. For a more mathematical consideration see section 2.1.2, also for a relation to different utility functions, introduced in section 1.3.2.

A.1 Preference Structure

In this section, we introduce properties of a preference structure which are usually assumed. We show how to derive a utility function from these properties and move forward the concept of maximizing expected utility of a terminal wealth over a whole wealth process.

We want to compare two investments A and B. Suppose A is at least as good as B, then we write:

$$A \succeq B$$

If B is in addition at least as good as A: $B \succeq A$, we write

$$A \sim B \text{ or } B \sim A$$

and say that we are indifferent between both investments and implicitly assume a symmetry property (from $A \sim B$ follows $B \sim A$ and vice versa). If $B \succeq A$ does not hold, then we strictly prefer A against B and write

$$A \succ B$$

We denote the set of all investments by $\mathbf{A}$. We assume the following properties:

Assumption A.1.1. *(Completeness) For all $A, B \in \mathbf{A}$, we have either $A \succeq B$ or $B \succeq A$.*

That means that we assume that the investor is not unsettled between any pair of investments in $\mathbf{A}$. This might be unrealistic, because we cannot expect that 80 years old man has any opinion to buy a skate board or inline skates when paying the same price. This does not mean that he cannot decide because he is indifferent between goods. We always consider money units so such examples can usually be ruled out. The next assumption excludes contradictions:

Assumption A.1.2. *(Transitivity) Let $A, B, C \in \mathbf{A}$, then*

$$A \preceq B, B \preceq C \Rightarrow A \preceq C.$$

Further we assume:

Assumption A.1.3. *(Reflexivity) For all $A \in \mathbf{A}$, we have*

$$A \preceq A$$

and

Assumption A.1.4. *(Continuity) For all $A \in \mathbf{A}$, we have that the sets $\{B \in \mathbf{A} : B \succeq A\}$ and $\{B \in \mathbf{A} : A \succeq B\}$ are closed.*

If assumption A.1.1, A.1.2, and A.1.3 (including symmetry) hold, one says that $\mathbf{A}$ induces a preference relation/structure $\succeq$. We know want to assign to every investment A a real number U(A), the utility of A. The absolute value of this function has no meaning. However, under the additional assumption (apart from A.1.1 and A.1.2) that for every investment A the set of investments weakly preferable to A is closed (assumption A.1.4), the following holds:

$$\exists U : \mathbf{A} \to \mathbb{R} \; \forall A, B \in \mathbf{A} : \; U(A) \geq U(B) \Leftrightarrow A \succeq B \tag{A.1}$$

see e.g. Varian (1994) Chapter 7. There are several other properties that are usually assumed, e.g. stability of preferences, that means in the considered time interval preferences do not change. In a pure investment problem it is assumed that if we have more units of one good this yields a higher preference as having less. The utility function is increasing. Further, if we have already five bikes the increase in utility of one more bike will be much less than the increase, if we do not have any bike. This leads to the concept of marginal utility. It is assumed to be decreasing. The utility function is concave. On the other hand this is a very nice property to mathematically work with utility functions. We can use convex analysis. Furthermore, U is often assumed to be continuously differentiable or even twice differentiable.

As seen in the introduction, it makes no sense to maximize the expected return, since we do not consider risk factors. Getting 3.5 is preferred against having 1 or 6 with equal probability, although the expected return is the same. The higher utility of six is not able to balance out the very low utility of one:

$$U(3.5) > \frac{1}{2}U(6) + \frac{1}{2}U(1)$$

The mathematical reason is the proposed concavity of U. This consideration leads to the concept of maximizing expected utility here

$$\max_{A,B} E(U(X)) = \max\{U(3.5), \frac{1}{2}U(6) + \frac{1}{2}U(1)\} = U(3.5),$$

where A a random variable attaining one or six with equal probability and B attains 3.5 with probability 1.

This hypotheses satisfies a lot of properties concerning rational behavior under uncertainty (four axioms, e.g. continuity and independence). However, empirical studies or even the above example (3.5 or 3.5 million) shows that in reality people do not always behave according to this hypotheses. One famous example is the St-Petersburg-Paradox: If an investor chooses his investments according to its expected value, then he would pay any amount of money to play the following game: We toss a coin, the investors gets \$2 if the dice shows tail, otherwise the dice is tossed again, if it is tail you get 2^2 and so on. So the expected value is infinity. But nobody would pay more than 20 or 30 dollars, even this is too much for this game, because it is very risky - very high outcomes are very rare and it is very probable getting say \$4 and paying \$20, who would do that? Hacking's (see Hacking (1982)) estimated that some people were even willing to pay \$25. One can try to handle this problem by saying the investor is risk-averse, having a logarithmic utility function ($\log x$). Then, the expected utility $E(\log B)$, where B is the described game, is about \$4. The investor plays if the stake is less than four dollars. But now suppose, the gain is $10^{(2^n)}$, then the expected utility is infinity as well, but would we play this game? We could move further by introducing an upper bound, but since this is not our topic, we refer the interested reader to a nice overview of this topic in Martin (2004). The idea - the so called St-Petersburg Paradox- goes back to Bernoulli (1738/1954), see Martin (2004) for further references.

There is a huge literature on this topic, from an empirical and theoretical point of view. We do not claim to be accurate or give a complete description of this topic. This section should only give a short hint, why it might make sense to optimize our portfolio according to the criterion of expected utility. There are other approaches, one short more mathematical consideration, especially concerned with claim pricing, can be found in Rouge and El Karoui (2000). The goal of this thesis is not to argue if this is a good approach or not. We will assume all rational behavior that is needed and take the criterion of expected utility as given throughout this thesis. The section was mainly taken from the lecture "Microeconomics I" given in summer 2001 at the University of Konstanz, (see the corresponding scriptum or Varian (1994)), and some common sense. Furthermore, there is a huge literature in this field, see e.g. Varian (1994) for further references.

A.2 Different Economic Problems

In this section, we describe different problems which can be modelled with the different utility functions introduced in section 1.3.2: A pure investment problem, a pricing problem together with equilibrium effects and a mixture of a pure investment and hedging problem:

Investment Problem

A reason to introduce a preference structure is to find an optimal - according to your preferences- investment strategy for a given initial wealth. We would like to maximize our expected wealth, but we do not want to bear too much risk while doing that. A portfolio with the same risk but higher return or with the same return and lower risk is preferred. Additionally, if we only concentrate on return, we obtain a higher degree of investor's satisfaction the higher the return. However, the higher the absolute return the less satisfaction is governed by an additional unit of return. An easy way to model a pure investment problem is to use power utilities with parameter $p \in (0, 1)$ or an exponential utility function. They are increasing and concave.

Pricing Problem and Equilibrium

As mentioned in section 1.2.1, in an incomplete market we are not able to price all contingent claims via a no-arbitrage argument, not every claim is attainable via a portfolio of the remaining stocks. Preferences of the investors facing risk have to be taken into account. Their attitude towards risk and return might be different among different investors with the same wealth and especially if it differs, e.g. between small and institutional investors. Returns of risky investments should be higher if investors in the economy are more risk-averse. In an equilibrium (supply equals demand), the price structure is therefore driven by the extend of risk-aversion of the investors in the market, see section 3.2.2 (Exponential Utility and Microeconomic Equilibrium). If we can replicate a price of an contingent claim by tradable assets the risk of these assets is in some sense transferred to the contingent claim. However, if we replicate a claim and also own the counterpart of the corresponding claim, we will not face risk. If replication is not possible with any initial capital, we cannot transform our pricing problem to a perfect hedging problem, we face risk. We wish to minimize this risk according to our preference structure, we try to meet the claim as close as possible instead. We set the desired price to the initial wealth necessary for the best approximate hedge. The pricing problem is now equivalent to an approximate hedging problem. Summarizing, if we are only interested in finding a price, then we will choose the price equal to the initial value of the process which is closest to the claim according to our preference structure, i.e. such that the utility of the difference between the final value of the process and the claim is maximized. We set the pricing problem equal to the problem of hedging a claim as close as possible, and the necessary initial wealth is then the price. In this case, we choose a model, which penalizes under- and overfitting. These leads to mean-variance hedging or more generally it is often modelled by isoelastic-utility functions with $p > 1$.

Investment and Hedging - Mixed Problem

Suppose you actually manage a company and you are interested in maximizing the general satisfaction of the stock holders. Goods have been bought and they need to be paid by a certain date T. Now, you have a certain amount of money and the goal is to find a strategy that meets the claim, but not at all costs. We are not interested in the price of the claim, we want to hedge it, i.e. be fairly safe that the claim can be paid, to maximize our over satisfaction in T. So investment of the remaining money does play a part. Getting a much higher wealth in some stages and slightly miss the claim in others might be preferred by the investor, since only a small punishment has to be paid in the slightly bad stages, but a large outperfoming of the claim gives a lot of utility in the good stages. Nevertheless, hedging the claim remains the main goal. Therefore, being away from the claim should be punished more extensively than rewarding when exceeding our aim by the same extend. We face an investment and a hedging problem. In the case, where we do not have enough money to hedge the claim a quadratic utility function is also used. But as soon as we also want to invest, a penalty for investing extra money - as in the mean variance model, is inappropriate. Also power utilities cannot be used because negative wealth are not possible or lead to a minus-infinity valued utility function. So this mixture ask for a certain shape of the utility function, e.g. the shape of an exponential utility function.

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1. Ich versichere hiermit, dass ich die vorliegende Arbeit mit dem Thema:

   Relation between L^q-Optimality, Exponential Control, and Entropy in Risk Management

   selbständig verfasst und keine anderen Hilfsmittel als die angegebenen benutzt habe. Die Stellen, die anderen Werken dem Wortlaut oder dem Sinne nach entnommen sind, habe ich in jedem einzelnen Falle durch Angaben der Quelle, auch der benutzten Sekundärliteratur, als Entlehnung kenntlich gemacht.

   Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

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