On Convergence to the Exponential Utility Problem

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Abstract

We provide a method to solve dynamic expected utility maximization problems with possibly not everywhere increasing utility functions in an L^p-semimartingale-setting. In particular, we solve the problem for utility functions of type $-e^{-x}$ (exponential problem) and $-(1-\frac{x}{2m})^{2m}$ (2m-th problem). The convergence of the 2m-th problems to the exponential one is proved. Using this result an explicit portfolio for the exponential problem is derived.

Key words: convex analysis, stochastic duality, exponential utility function, minimal entropy martingale measure, convergence of q-optimal martingale measures, wealth and portfolios
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1 Introduction

Besides the control-theoretical interest there is an economic motivation for the use of exponential utility functions. Optimizing the investment decisions for a certain time horizon T of an investor with initial wealth x can be described

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by maximizing the expected exponential utility of a terminal value Y_T of a wealth process $Y = x + \tilde{Y}$:

$$V_{exp,\xi}(x) = \max E(1 - e^{-(x+\tilde{Y}_T-\xi)}), \tag{1}$$

where ξ represents a financial obligation the investor faces in T. In a semimartingale model the problem of finding an optimal terminal value of the exponential problem was completely solved, including contingent claims ξ, in Delbaen et al. (2002) and Kabanov and Stricker (2002), for different classes of wealth processes. Variants of the concept appeared before, see Remark 2.1 in Delbaen et al. (2002) for further references. Moreover, a backward stochastic differential equation (BSDE) approach is found in Rouge and El Karoui (2000) and Hu et al. (2005). The second article avoids dual relations and applies martingale properties of the value function, instead.
More generally, Schachermayer (2001) completely solved the utility maximization for a wider class of utility functions. However, these approaches do not cover not-everywhere-increasing utility functions. Furthermore, the explicit form of the optimal portfolio has not been derived, except in very special cases for the exponential utility problem, see e.g. Rouge and El Karoui (2000) and Delbaen et al. (2002). On the other hand, for isoelastic utility functions with parameter $\alpha > 1$ explicit portfolios are known. We therefore present a complete relation between various types of martingale measures (dual problem), the iso-elastic, and the exponential problem (Theorem 7). This new approach contains convergence of the terminal values leading to an explicit portfolio of the exponential problem. We further propose a method to solve dynamic utility maximization problems for possibly not everywhere increasing utility functions.
As we consider p-integrable strategies (see Definition 2.1), terminal values of allowed wealth processes are elements of L^p. We reformulate the dynamic optimization problem over wealth processes as a constraint static problem over L^p-random variables. This is implicitly done for increasing utility functions in Delbaen and Schachermayer (1996b). We present an extension of this result also applicable to not-everywhere-increasing, concave utility functions. For the same class of functions, we further suggest a method to solve the constraint problem using results from convex analysis. In particular, we obtain the optimal solution for utility functions of the form: $-(1 - \frac{x}{2m})^{2m}$. The optimal terminal value turns out to be a function of the $\frac{2m}{2m-1}$-optimal martingale measure, which is the solution of the corresponding dual problem from convex analysis. It is known that the q-optimal martingale measure converges to the minimal entropy measure – up to a scaling constant the optimal solution of the dual exponential problem. We use these results to show that the terminal values and the value functions of the utility problem corresponding to the sequence $(-(1 - \frac{x}{2m})^{2m})_m$ converge to the terminal value of the exponential utility function. This convergence then yields the convergence of the portfolios and provides an explicit portfolio for the exponential utility problem in the

same setting with a deterministic terminal trade-off. Extensions are possible, but rather technical and go beyond the scope of this paper. Further note that proofs are given in the case of a trivial claim $\xi \equiv 0$. Fortunately, results remain valid in the non-trivial case leading to an interesting result, see remark 11.

The paper is organized as follows: In section 2, we explain the market model with L^p-strategies and formulate the main problem in a dynamic and a static version. Using techniques from convex analysis, section 3 describes a method to solve the dual problem, a constraint static problem. We therefore cite some results on different possible dual solutions - the minimal entropy martingale measure, the minimal martingale measure, and q-optimal martingale measures for $q > 1$ in section 3.2. Using these results, we derive the main result of this paper in section 4: The convergence of the terminal values and the value functions of the 2m-th problems to the corresponding values and functions in the exponential problem. As an application, section 5 gives the corresponding convergence result of the portfolios and presents the optimal portfolio in the exponential case.
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2 The Market Model and Problem Formulation

We work with a semimartingale model: Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T \in (0, \infty)$ a time horizon, and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ a filtration satisfying the usual conditions, i.e. right-continuity and completeness. This enables us to use right-continuous with left-limits (RCLL) versions for all $(P, \mathbb{F})-$semimartingales representing our stocks. As only special semimartingales are considered, so that a Doob-Meyer- decomposition holds, we simply call them semimartingales. All expectations and spaces without a subscript are defined with respect to the measure P. K denotes a generic positive constant. Throughout this paper a continuous $\mathbb{R}^{n+1}-$valued $(P, \mathbb{F})-$semimartingale $(S, 1)$ is given, where $S = (S_t)_{t\in[0,T]}$ with unique decomposition $S = S_0 + M + A$ into a local martingale M and a predictable process of bounded variation A. S represents a vector of n risky assets and 1 stands for a riskless asset with constant discounted price, i.e. the riskless asset serves as a numéraire. Moreover, a self-financing strategy (x, N) is then given by the initial wealth x and the number of shares $N = (N^1, ..., N^n)$ of the stocks held at time $t \in [0, T]$. We require that our strategies are predictable and satisfy an integrability condition:

Definition 2.1 *The set of L^p-trading or p-integrable strategies is defined as* *follows:*

$$\mathcal{A}^p := L^p(M) \cap L^p(A)$$

where

$$L^p(M) = \{N \in \boldsymbol{P}^n |\ \|N\|_{L^p(M)} < \infty\},\ \ L^p(A) = \{N \in \boldsymbol{P}^n |\ \|N\|_{L^p(A)} < \infty\}$$

with $\|N\|_{L^p(M)} := \|(\int_0^T Nd\langle M\rangle_t N')^{\frac{1}{2}}\|_{L^p}$, $\|N\|_{L^p(A)} := \|\int_0^T |N||dA_t|\|_{L^p}$ *and* $\boldsymbol{P}^n$ *the set of all predictable* $\mathbb{R}^n$*-valued processes.*

See Jacod (1979) or Protter (2004) for undefined notations and the standard theorems concerning the theory of integration with respect to semimartingales. Self-financing strategies in $\mathcal{A}^p$ then define a wealth process $x + \int_0^t NdS$ for $t \in [0, T]$. The integrability assumption implies that the set of terminal values of allowable wealth processes is a subset of $L^p(P)$:

$$\mathcal{G}^p(x) := \{Y_T | Y \in \mathcal{W}(x)\} \subset L^p(P) \tag{2}$$

where $\mathcal{W}(x) := \{Y | Y_t = x + \int_0^t NdS, N \in \mathcal{A}^p\} \subset \mathcal{H}^p(P)$ is the class of all wealth process generated by the class of L^p-trading strategies, i.e. Y_T can be hedged by the initial wealth $x \in \mathbb{R}$ and a trading strategy N. (For a definition of $\mathcal{H}^p$ see Protter (2004), shortly it is the space of all canonical decomposition $S = M + A + S_0$ such that $\|S\|_{\mathcal{H}^p} := \|[M]_T^{1/2} + \int |dA_s|\|_{L^p} + \|S_0\|_{L^p}$ is finite). The chosen class excludes doubling strategies by the Burkholder-Davis-Gundy-inequality. To exclude arbitrage opportunities (note: here the no arbitrage notion is the notion of no free lunch with vanishing risk) we assume that the space of all equivalent martingale measures with L^q-integrable densities is nonempty, i.e. $\mathcal{M}_e^q \neq \emptyset$, where

$$\mathcal{M}_e^q = \{Q | dQ = Z_T dP,\ Z \in \mathcal{D}_e^q\} \subset L^q(P),\ \frac{1}{p} + \frac{1}{q} = 1$$

with $\mathcal{D}_e^q = \{Z \in \mathcal{U}^q | E(Z_T) = 1,\ Z_T > 0,\ SZ \in \mathrm{M}_{loc}\}$ and $\mathcal{U}^q$ is the class of uniformly integrable martingales M with $E^{\frac{1}{q}}(|M_T|^q) < \infty$. We add a subscript Z when densities are meant, e.g. $\mathcal{M}_{e,Z}^q$. Spaces with subscript a instead of e only require $Z_T \geq 0$ and whereas spaces like $\mathcal{M}_S$ denote the class of signed local martingale measure, i.e. Z_T does not have to be non-negative. If $\mathcal{M}_e^q$ is a singleton, we call the market complete, otherwise incomplete. When the notation is clear from the context, we write Z instead of Z_T and add a superscript to Z when denoting a density process.
Before concluding this section, we come back to the set of allowable trading strategies. Delbaen and Schachermayer (1996b) consider simple p-admissible strategies and define the corresponding integral. Since S is assumed to be continuous and $\mathcal{M}_e^q \neq \emptyset$, the closure of the space of these integrals $\mathbb{K}^p(x)$ is equal to the closure of $\mathcal{G}^p(x)$, see Lemma 2.1 in Grandits and Rheinländer (2002a). For $\mathbb{K}^p(x)$, Delbaen and Schachermayer (1996b) show a hedging result for L^p claims, which then also holds for $\mathcal{G}^p(x)$, if it is already closed. The closedness is

true under assumption 2.1 (Reverse Hölder inequality, see Theorem Grandits and Krawczyk (1998) Theorem 4.1). So using this assumption, we have the hedging result mentioned above:
Every $f \in L^p$ satisfying $E_Q(f) = x$ for every $Q \in \mathcal{M}_a^q$ is in $\mathcal{G}^p(x)$ or f can be replicated with initial wealth x, respectively.
We start by introducing the Reverse Hölder inequality $R_q(Q)$:

Definition 2.2 *A process Z satisfies the Reverse Hölder inequality $R_q(Q)$, if there exists a $K(q) > 1$ such that*

$$\sup_{\tau \in \mathcal{T}} E_Q \left(\left| \frac{Z_T}{Z_\tau} \right|^q |F_\tau \right) < K(q). \tag{3}$$

where $\mathcal{T}$ is the class of stopping times $\tau \leq T$.

$\mathcal{M}_e^q \neq \emptyset$ for some $q > 1$ is then a consequence of the following stronger assumption used in Santacroce (2005):

Assumption 2.1 *A) All $(F, P)-$ local martingales are continuous.*
B) There exists an equivalent martingale measure Q such that its density process satisfies the reverse Hölder inequality $R_{q_0}(P)$ for some fixed $q_0 > 1$.

Under this assumption the unique solution of $\min_{Q \in \mathcal{M}_S^q} E((\frac{dQ}{dP})^q)$ exists in $\mathcal{M}_e^q$. It is called the q-optimal martingale measure Q_q. Moreover, the density process of Q_q, denoted by $Z^{(q)}$, satisfies the reverse Hölder inequality $R_{q_0}(P)$ for some fixed $q_0 > 1$, if assumption 2.1 B) holds and S is continuous (see Theorem 4.1 Grandits and Krawczyk (1998)).

To include non-increasing utility functions, we extend the class of wealth processes to

$$\mathcal{W}_C(x) = \{Y | Y_t = x + \int_0^t N dS - C_t, N \in \mathcal{A}^p, \ C \in \mathcal{K}^p\}$$

where $\mathcal{K}^p$ the class of increasing right-continuous processes with $\int_0^T |dC_t| \leq C_T \in L^p$. Note, $\mathcal{W}_C$ is a subset of the set of p-integrable wealth processes.
We consider the following dynamic optimization problem:

$$V(x)_{\xi,C} \equiv \sup_{Y \in \mathcal{W}_C(x)} E[U(Y_T - \xi)], \ x \in \mathbb{R} \tag{4}$$

where U is a concave, not necessarily increasing function, ξ an $\mathcal{F}_T$-measurable, L^p-integrable random variable, and $E[U(X - \xi)] < \infty$. From a proof analogous to the one of Theorem 2.1.1 in El Karoui and Quenez (1995) $J_t = \operatorname{ess\,sup}_{Q \in \mathcal{M}_e^q} E_Q(X|\mathcal{F}_t)$ is a right-continuous Q-supermartingale for every $Q \in \mathcal{M}_e^q$. By the optional decomposition Theorem in Föllmer and Kabanov (1998) and some very technical estimations, J is in $\mathcal{W}_C(\sup_{Q \in \mathcal{M}_e^q} E_Q(X))$. Hence, if

$E[U(X-\xi)] < \infty$ problem (4) is equivalent to the following static problem:

$$V(x)_{\xi,C} \equiv \sup_{X\in L^p(\mathcal{F}_T),\forall Q\in\mathcal{M}_e^q E_Q(X)\le x} E[U(X-\xi)],\ x\in\mathbb{R}. \tag{5}$$

As mentioned above, a proof for increasing utility functions can be found in Delbaen and Schachermayer (1996b). In the sequel, we tackle the static problem using methods from convex analysis. We explicitly solve the problem for the exponential utility function $U_{\exp}(x) = -e^{-\alpha x},\ \alpha > 0$ and its - not everywhere increasing - approximating sequence $U_{2m}(x) = -(1-\frac{\alpha x}{2m})^{2m}$. We show that their solutions converge. Note that for simplicity we set $\alpha = 1$, a generalization is straightforward.

Remark 1 *(i) We use the following notation throughout the paper: For $m \in \mathbb{N}$, we let $p = 2m$ and hence from $\frac{1}{q}+\frac{1}{p} = 1$ we have $q = \frac{2m}{2m-1} \to 1,\ m \to \infty$. So when the notation seems more convenient and unambiguous, we write $Z^{(q)} = Z^{(2m)}$ for processes used in the $q = \frac{2m}{2m-1}$ situation for the $2m$-problem and $Z_q = Z_{2m} := Z_T^{(2m)}$ for its terminal values.*
(ii) The continuity assumption of S is not necessarily needed to solve the 2m-th problem. However, the q-optimal martingale measure is only proved to be a signed local martingale measure, i.e. in $\mathcal{M}_S^q$ see e.g. Leitner (2001). Further, the reformulation of the dynamic to the static 2m-th problem becomes a bit more complicated. Since we need continuity for our convergence result, we stick to this continuity of S throughout the paper.

Remark 2 *We consider the exponential utility problem as a limit of the $2m$-problems. So the setting of the exponential problem may be derived from the specific convergence properties derived below.*
On the other hand, we could also ***define*** *the setting of the exponential control problem and then derive the desired properties from our convergence results. A sufficient set of assumptions should imply that the gains process is bounded from below and its martingale part should be a BMO-martingale. The main additional assumption in a Brownian setting would be to bound the full rank part of σ away from 0 and ∞. For details, see e.g. Hu et al. (2005) and for the properties of BMO-martingales Revuz and Yor (1991) and Meyer (1976). However, we would lose some of the generality of the approach here. In section 5 we therefore derive a portfolio in the case of a deterministic terminal value of the trade off process (see the definition below). This strategy is contained in all mentioned spaces, but even more it is an element of the following quite canonical space extending the class of L^p-trading strategies:*
$\mathcal{A}^{exp} = \{N \in \bigcap_{p>1}\mathcal{A}^p : Ee^{-\alpha\int_0^T NdS} < \infty\}$.

3 Solving Static Utility Optimization Problems

3.1 General Approach

Using Theorem 2 in Luenberger (1969) (p.221), we obtain:

Corollary 3 *Suppose there exists a* $y_0 \in \mathbb{R}_0^+$, *a* $Z_{opt} \in \mathcal{M}_{a,Z}^q$, *and an* $X_0 \in \mathcal{O}^p := \{X \in L^p(P) : EU(X) < \infty\}$ *such that the Lagrangian* $\tilde{L}(X, y \cdot Z) := E(U(X)) - y(E(ZX) - x)$ *possesses a saddle point at* $(X_0, y_0 \cdot Z_{opt})$, *i.e.*

$$\tilde{L}(X_0, y \cdot Z) \geq \tilde{L}(X_0, y_0 \cdot Z_{opt}) \geq \tilde{L}(X, y_0 \cdot Z_{opt}),$$

for all $X \in \mathcal{O}^p$, $y \in \mathbb{R}_0^+, Z \in \mathcal{M}_{a,Z}^q$ *or* $yZ = \lambda^* \in D := \mathbb{R}_0^+ \times \mathcal{M}_{a,Z}^q$, *respectively. Then* X_0 *solves:*

$$\max E(U(X)),\ s.t. \forall Q \in \mathcal{M}_a^q :\ E_Q(X) \leq x,\ X \in \mathcal{O}^p \tag{6}$$

■

For a proof, let $\mathcal{P} := \{X \in L^p(P) : \forall Z \in \mathcal{M}_{a,Z}^q,\ y \geq 0,\ (X, F_{y,Z}) := E(yZX) \geq 0\}$ where $(\cdot,\cdot)$ denotes the obvious dual pairing, and so $D := \mathcal{P}^{\oplus} = \{\lambda^* :\ \lambda^* = y \cdot Z,\ y \in \mathbb{R}_0^+,\ Z \in \mathcal{M}_{a,Z}^q\}$, $G(X) = X - x$, and $h(X) = E(U(X))$ in Theorem 2 Luenberger (1969).

Before proving existence in the exponential and the $2m$-cases (Theorem 6), we give a method to search for a saddle point of the Lagrangian $\tilde{L}$ in the abstract setting given above. The proof is then given by applying this method and proving the necessary integrability conditions. We start treating the second inequality of Corollary 3: $X_0(\lambda_1^*) = \arg\max_X \tilde{L}(X, \lambda_1^*)$ for an arbitrary λ_1^*. From the Lagrangian, the convex dual $\check{U}(y) := \sup_{x \in \mathbb{D}}[U(x) - xy]$ canonically arises. $\mathbb{D}$ denotes the domain of U. If U is strictly concave and continuously differentiable - not necessarily increasing - then $\check{U}(y) = U(I(y)) - I(y)y$, where $I := (U')^{-1}$. The minimizer $I(y)$ is unique. And so for a fixed $\lambda_1^* = y_1 Z_1$ and all $X \in L^p$:

$$\tilde{L}(X, \lambda_1^*) \leq E(\check{U}(\lambda_1^*)) + xy_1,$$

equality holds if and only if $X_0(\lambda_1^*) = I(\lambda_1^*) = I(Z_1 \cdot y_1)$. The problem of finding a λ_0^* that also satisfies the first inequality, i.e. $\forall \lambda^* \in D :\ \tilde{L}(X_0(\lambda^*), \lambda^*) \geq \tilde{L}(X_0(\lambda_0^*), \lambda_0^*)$, is equivalent to the following dual problem:

$$\min_{y_1 \geq 0, Z_1} \phi(y_1, Z_1) \tag{7}$$

where $\phi(y_1, Z_1) = E(U(I(Z_1 \cdot y_1)) - Z_1 \cdot y_1 I(Z_1 \cdot y_1)) + xy_1$. In the sequel, $\lambda_0^* = y_0 Z_{opt}$ denotes the optimal solution of (7). So $(\lambda_0^*, X_0(\lambda_0^*))$ is a saddle point, provided that $X_0(\lambda_0^*) \in \mathcal{O}^p$. Hence,

$$X_0(\lambda_0^*) = I(y_0 Z_{opt}) \in \mathcal{O}^p \tag{8}$$

is the optimal solution of the primal problem. Suppose the dual solution exists. To explicitly solve the dual problem, we perform a second minimization: $Z(y_0) = \arg\min_{Z \in \mathcal{M}_a^q} \phi(y_0, Z)$. Putting this into the dual problem, the dual solution is either $(0, Z(0))$ or $(y_0, Z(y_0))$, where y_0 the solution of

$$\mathcal{X}_{Z(y_0)}(y_0) = E(Z(y_0) I(Z(y_0) y_0)) = x. \tag{9}$$

Denote the unique solution of (9) by $\mathcal{Y}(x)$. It turns out that for large enough m (dependent on x) the optimization problem of $-(1 - \frac{x}{2m})^{2m}$ and the exponential utility function is independent of the initial wealth. $\mathcal{Y}(x)$ exists and is positive. So the solution of (9) in the case of the 2m-th ($\mathcal{Y}_{2m}$) and the exponential problem ($\mathcal{Y}_{exp}$) can easily be derived by inverting $\mathcal{X}_{2m}$ resp. $\mathcal{X}_{exp}$. This leads to the solutions of the dual problems: The $\mathcal{Y}_{2m}(x)$ times $\frac{2m}{2m-1}-$optimal martingale measure and $\mathcal{Y}_{\exp}(x)$ times the minimal entropy martingale measure, respectively.

3.2 q-optimal Martingale Measures and the Minimal Entropy Martingale Measure

The term relative entropy is used in information theory. One looks for a martingale measure that -in an intuitive sense- carries most information about P:

$$Q_{min} = \arg \min_{Q \in \mathbb{P}_f(P)} H(Q|P),$$

where $\mathbb{P}_f(P) := \{Q \in \mathcal{M}_a : H(Q|P) < \infty)\}$ with $H(Q|P) = E_P(\frac{dQ}{dP} \log(\frac{dQ}{dP}))$ if $Q \ll P$ and ∞ otherwise. If $\mathbb{P}_f(P) \neq \emptyset$, the unique existence follows from Theorem 2.1. in Frittelli (2000). If in addition $\mathbb{P}_{f,e}(P) := \mathcal{M}_e \cap \mathbb{P}_f(P) \neq \emptyset$, then $Q_{min} \in \mathcal{M}_e$, i.e. Q_{min} is equivalent to P (Theorem 2.2.). Q_{min} is known as *the minimal entropy martingale measure.*

By assumption 2.1, S is continuous and therefore satisfies the structure condition and admits the decomposition $S = S_0 + M + \int d\langle M \rangle \hat{\lambda}$, where M a continuous local martingale and $\hat{\lambda}$ a predictable $\mathbb{R}^n-$valued process, as defined in Schweizer (1995). The process $\hat{K} = \langle -\int \hat{\lambda}' dM \rangle = \int \hat{\lambda}' d\langle M \rangle \hat{\lambda}$ is called the *mean-variance trade-off process.* If the Doléans-Dade exponential $\hat{Z} = \mathcal{E}(-\int \lambda dM)$ is a martingale, *the minimal martingale measure* is defined as

$$d\hat{Q} = \hat{Z}_T dP.$$

For a definition offering more interpretation in the original case, we refer to Föllmer and Schweizer (1991).

The minimal entropy martingale measure can be described by a backward stochastic differential equation (BSDE hereafter). From Theorem 1 in Schweizer (1995), we know that every equivalent martingale measure can be represented as $\frac{dQ}{dP} = Z_Q,\ Z_Q = \mathcal{E}_T(M^Q), M^Q \in \mathbb{M}_{loc}$. Further, using the notation $\mathcal{E}_{tT}(M^Q) = \frac{\mathcal{E}_T(M^Q)}{\mathcal{E}_t(M^Q)}$, Mania et al. (2003b) prove the following characterization of the minimal entropy martingale measure (Theorem 3.1.):

Theorem 4 *Let all $(\mathcal{F}, P)$ local martingales be continuous and $\mathbb{P}_{f,e}(P) \neq \emptyset$. Then the value process V_t, given by*

$$V_t = \text{ess} \inf_{Q \in \mathbb{P}_{f,e}(P)} E_Q(\log \mathcal{E}_{tT}(M^Q)|\mathcal{F}_t),$$

is a special semimartingale with $V_t = m_t + A_t + V_0$, where $m \in \mathbb{M}^2_{loc}$, ($\mathbb{M}^2_{(loc)}$ denotes the space of all (local) martingales M with $\| \sup_t M_t^2 \|_{L^1} < \infty$) and A a locally bounded variation predictable process. Therefore the Galtchouk-Kunita-Watanabe (G-K-W) decomposition exists: $m_t = \int_0^t \phi'_s dM_s + \tilde{m}_t$, $\langle \tilde{m}, M \rangle = 0$. Furthermore V_t is the solution of the following BSDE:

$$Y_t = Y_0 - \text{ess} \inf_{Q \in P_{f,e}(P)} (\frac{1}{2}\langle M^Q \rangle_t + \langle M^Q, L \rangle_t) + L_t,\ Y_T = 0 \tag{10}$$

Moreover, Q_{min} is the minimal entropy martingale measure if and only if

$$\frac{dQ_{min}}{dP} = \mathcal{E}_T(M^{Q_{min}}),\ M_t^{Q_{min}} = -\int_0^t \hat{\lambda}'_s dM_s - \tilde{m}_t \tag{11}$$

Suppose, in addition, the minimal martingale measure exists, i.e. $\hat{Z}$ is a martingale, and satisfies the Log-Reverse-Hölder-inequality, for a definition see e.g. Grandits and Rheinländer (2002a). Then, V uniquely solves the above BSDE (10) and is bounded.

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A similar characterization is proven for the q-optimal martingale measure in Mania et al. (2003a):

Theorem 5 *If $\mathcal{M}_q^e \neq \emptyset$ and all $P-$local martingales are continuous, then the following assertions are equivalent:*

(1) the martingale measure Q_q is q-optimal
(2) Q_q is a martingale measure satisfying

$$dQ_q = \mathcal{E}_T(M^{Q_q})dP, \tag{12}$$

where
$$M_t^{Q_q} = -\int_0^t \hat{\lambda}'_s dM_s - \frac{1}{q-1}\int_0^t \frac{1}{V_s(q)} d\tilde{m}_s(q) \tag{13}$$

$V(q)_t = V_0(q) + m(q)_t + A(q)_t$ *is equal to* $\operatorname{ess\,inf}_{Q\in\mathcal{M}_e^q} E((\mathcal{E}_{tT}(M^Q))^q|\mathcal{F}_t)$, *it uniquely solves the following BSDE:*

$$Y_t = Y_0 - \text{ess}\inf_{M^Q\in\mathcal{M}_q^e}[\frac{1}{2}q(q-1)\int_0^t Y_s d\langle M^Q\rangle_s + q\langle M^Q, L\rangle_t] + L_t,\ t<T,$$
$$Y_T = 1.$$

$\tilde{m}(q)$ *denotes the orthogonal part of the G-K-W-decomposition of* $m(q)$*:*

$$m_t(q) = \int_0^t \phi'_s(q)dM_s + \tilde{m}_t(q) \tag{14}$$

If $\mathcal{E}(-\int_0^t \hat{\lambda}'_s dM_s)$ *is a martingale, i.e. the minimal martingale measure exists and in addition it satisfies the Reverse Hölder condition, then the value process* $V(q)$ *above is the unique solution of the above BSDE and there exist positive constants* k *and* K *such that almost surely for all* $t\in[0,T]$ *:*

$$k \le V_t(q) \le K$$

■

A simple consequence of two Corollaries of Theorem 4 and Theorem 5, Corollary 3.4 in Mania et al. (2003b) and Corollary 3 in Mania et al. (2003a) (also see Santacroce (2005)), is that if $\hat{K}_T$ is deterministic the minimal entropy martingale measure, the minimal martingale measure, and the $q-$optimal martingale measures $q>1$ coincide almost surely. Under the weaker assumption 2.1, Santacroce (2005) establish that:

$$E(\langle\frac{m(q)}{q-1} - m\rangle_T) \to 0,\ q\downarrow 1 \tag{15}$$

Furthermore,
$$E\sup_{t\le T}|Z_t^{(q)} - Z_t^{min}| \to 0,\ q\downarrow 1, \tag{16}$$

in particular, $Z_T^{(q)} \overset{L^1}{\to} Z_T^{min}$, $q\downarrow 1$, where $(Z_t^{(q)})_t$ and $(Z_t^{(min)})_t$ are density processes of the $q-$optimal martingale measures and the minimal entropy martingale measure, respectively. The last assertion, using a duality approach, is also proven in Grandits and Rheinländer (2002a). Assumptions are more or less the same, the obtained convergence is weaker.

Next, we see that the dual solution of the optimization problem with utility function $-(1-\frac{x}{2m})^{2m}$ is the $\frac{2m}{2m-1}-$optimal martingale measure times $\mathcal{Y}_{2m}(x)$

and the dual of the exponential problem is the minimal entropy martingale measure times $\mathcal{Y}_{\exp}(x)$. So the above considerations already show that the dual measures converge.

3.3 Exponential Utility Function and its Approximating Sequence

Using the above approach, we solve the static problem given in (6) with $U(X) = -e^{-\alpha X}$ and an arbitrary $p > 1$. Although our static problem is quite general, we can show that the optimal value $X_0^{(exp)}$ coincides with the usual optimal terminal value of the dynamic exponential problem characterized e.g. in Delbaen et al. (2002) and Kabanov and Stricker (2002). In contrary to the quite restricted classes of strategies in these papers, our approach leaves much space to define a wide class of portfolios, e.g. $\mathcal{A}^{exp}$. In section 5, the optimal $X_0^{(exp)}$ will turn out to be the limit of the optimal solutions of the $2m - th$ problem. Using this, we give the problem a dynamic component by developing, under some weak assumptions, a strategy that reaches $X_0^{(exp)}$. In the sequel and if it is clear from the context, we denote by X_0 the optimal solution of the considered optimization problem, e.g. $X_0 = X^{(exp)}$.
Without loss of generality we set $\alpha = 1$. By (8) we obtain:

$$X_0(Z_0 y_0) = I(Z_0 y_0) = -\log(Z_0 y_0) \tag{17}$$

where $(Z_0 \cdot y_0)$ is the minimizer of

$$\begin{aligned}\phi(y_0, Z_0) &= E(U(I(Z_0 \cdot y_0)) - Z_0 \cdot y_0 I(Z_0 \cdot y_0)) + x y_0 \\ &= -y_0 + x y_0 + y_0 \log y_0 + y_0 H(Q_0|P)\end{aligned}$$

with $Z_0 = \frac{dQ_0}{dP}$. We have $y_0 \geq 0$, so as above, we start deriving $Z(y_0)$:

$$Z(y_0) = \arg\min_Z \phi(y_0, Z).$$

Clearly, $Z(y_0)$ is equal to the density related to the minimal entropy martingale measure $Q_{min} = \arg\min_Q H(Q|P)$ and independent of y_0, therefore also independent of the initial wealth x. To determine y_0, we apply the result in equation (9), i.e. $\mathcal{X}_{Z(y_0)}(y_0) = x$:

$$\begin{aligned}\mathcal{X}_{Z(y_0)}(y_0) &= \mathcal{X}_{Z_{min}}(y_0) = E(Z_{min} I(Z_{min} y_0)) \\ &= E(Z_{min}(-\log(Z_{min} y_0))) = -E(Z_{min} \log Z_{min}) - \log y_0 \\ &= -H(Q_{min}|P) - \log y_0\end{aligned}$$

We calculate the inverse of $\mathcal{X}$ and finally obtain the solution:

$$\begin{aligned}\mathcal{Y}(x) &= \exp\{-H(Q_{min}|P) - x\} \\ X_0(x) &= X_0(\mathcal{Y}(x)) = I(Z_{min}\mathcal{Y}(x)) = -\log Z_{min} + H(Q_{min}|P) + x \qquad (18)\end{aligned}$$

By plugging the optimal solution into $\sup_{X\in\mathcal{O}^p} E(-e^{-X})$, we obtain a duality under an arbitrary probability measure P with $\mathbb{P}_{f,e}(P) \neq \emptyset$:

$$\sup_{X\in\mathcal{O}^p} E(-e^{-X}) = -e^{-x-\min_{Q\in\mathbb{P}_{f,e}(P)} H(Q|P)} \qquad (19)$$

Note, we still have to prove that $X_0 \in \mathcal{O}^p$ for all $p > 1$. Under the assumptions of Theorem 4, we have $E((M^{Q_{min}})^{2p}_T) < \infty$, hence:

$$\begin{aligned}E\log^p Z_{min} &= E(M^{Q_{min}}_T - \frac{1}{2}[M^{Q_{min}}]_T)^p \\ &\leq 2^{p-1}E((M^{Q_{min}}_T)^p + (\frac{1}{2})^p[M^{Q_{min}}]^p_T) \\ &\leq K(p)\cdot E((M^{Q_{min}}_T)^{2p}) < \infty \qquad (20)\end{aligned}$$

by the inequalities of Burkholder-Davis-Gundy and Doob and $K(p)$ a positive constant dependent on p. Further, $E(e^{-X_0^{(exp)}}) = e^{-H(Q_{min}|P)-x} < \infty$.

We turn to the solution of the 2m-th problem (w.r.t the utility function $u_{2m}(x) = -(1-\frac{x}{2m})^{2m}$): The strictly monotonic derivative of u_{2m} is $(1-\frac{x}{2m})^{2m-1}$. So we have:

$$I_{2m}(y) := (u')^{-1}_{2m}(y) = (1-y^{\frac{1}{2m-1}})2m,$$

and the dual problem, described in (7) converts to:

$$\min_{y\in\mathbb{R}_0^+, Z\in\mathcal{M}^{\frac{2m}{2m-1}}_{a,Z}} (2m-1)y^{\frac{2m}{2m-1}}E(Z^{\frac{2m}{2m-1}}) - 2myE(Z),$$

which has the same solution as $\min_{Z\in\mathcal{M}^{\frac{2m}{2m-1}}_{a,Z}} E(Z^{\frac{2m}{2m-1}})$, the $\frac{2m}{2m-1}$-optimal martingale measure. Recall, we denote its density process by $Z^{(2m)}$ and the related density by $Z^{(2m)}_T =: Z_{2m}$. It is independent of y. So

$$\mathcal{X}_{Z_{2m},2m}(y) = E(Z_{2m}I_{2m}(yZ_{2m})) = 2m - 2my^{\frac{1}{2m-1}}E(Z_{2m}^{\frac{2m}{2m-1}})$$

Consequently:

$$\mathcal{Y}_{Z_{(2m)},2m}(x) = \left((2m-x)\left(2mE\left(Z_{2m}^{\frac{2m}{2m-1}}\right)\right)^{-1}\right)^{2m-1} \qquad (21)$$

and

$$X_0^{(2m)}(x) = 2m - 2m\left(Z_{2m}\left(\frac{2m-x}{2mE\left(Z_{2m}^{\frac{2m}{2m-1}}\right)}\right)^{2m-1}\right)^{\frac{1}{2m-1}} \tag{22}$$

Finally, we have to check whether $X_0^{(2m)}$ is in L^{2m}. This is clear from $|X_0^{(2m)}(x)|^{2m} \leq K_x(m)|Z_{2m}|^{\frac{2m}{2m-1}} < \infty$ as $Z_{2m} \in L^q$, where $q = \frac{2m}{2m-1}$ and $K_x(m)$ is a constant depending on m and x.
Summarizing, we thus have

Theorem 6 *If $\mathcal{M}_e^q \neq \emptyset$, the Lagrangian in Corollary 3 with $U(x) = -(1 - \frac{x}{2m})^{2m}$ possesses a saddle point at $(X_0^{(2m)}(x), \mathcal{Y}_{2m}(x)Z_{2m})$. The corresponding $2m$-th static problem (6) has a solution (see (22)).*
If $U(x) = \exp(-x)$, under the assumptions of Theorem 4, a saddle point exists and is given by $(X_0^{exp}(x), \mathcal{Y}_{exp}(x)Z_{min})$, where $X_0^{exp}(x)$ (see 18) is the solution of the static exponential problem.

4 Convergence of the Terminal Values and the Value Functions

This section is devoted to the convergence of the terminal values and the value functions of the $2m$-th problem to the exponential one. After some estimations, the fact that $I_{2m}(y) = 2m(1 - y^{\frac{1}{2m-1}})$ converges to $I_{\exp}(y) = -\log y$, and the convergence of the $\frac{2m}{2m-1}$-optimal measures to the minimal entropy martingale measure yield $Z_{2m}I_{2m}(y_mZ_{2m}) \xrightarrow{P} Z_{min}I(yZ_{min})$ for an arbitrary real sequence $(y_m)_m$ with limit y. After establishing this, we show that $\mathcal{Y}_{Z_{2m}(2m)}(x)$ converges to $\mathcal{Y}_{Z_{min},exp}(x)$ or equivalently their corresponding inverse functional $\mathcal{X}$, for large enough m (to ensure that $\mathcal{Y}_{2m}(x)$ is strictly positive). Together, this yields:

$$X_0^{(2m)}(x) = I_{2m}(Z_{2m}\mathcal{Y}_{2m}(x)) \xrightarrow{P/a.s.} I_{exp}(Z_{min}\mathcal{Y}_{exp}(x)) = X_0^{exp}(x)$$

Note, the kind of convergence depends on the given assumptions and is specified later. Convergence in probability can be strengthened by establishing uniform integrability of $(I_{2m}(Z_{2m}\mathcal{Y}_{2m}(x)))_m$. Establishing all these steps yields our main theorem:

Theorem 7 *In our model with an $\mathcal{F}_t$-adapted continuous semimartingale $S = S_0 + M + A$, let one of the following assumptions be satisfied:*

(1) Assumption 2.1
(2) The terminal value of the mean variance tradeoff process ($\hat{K}_T = \langle -\int \hat{\lambda}' dM\rangle_T$) is deterministic.

Then, the solution of 2m-th problem converges in L^1 to the solution of the exponential problem, i.e.:

$$\begin{aligned} X_0^{(2m)}(x) &= 2m - 2m\left(Z_{2m}\left(\frac{2m-x}{2mE\left(Z_{2m}^{\frac{2m}{2m-1}}\right)}\right)^{2m-1}\right)^{\frac{1}{2m-1}} \\ &\stackrel{L^1}{\longrightarrow} X_0^{exp}(x) = -\log Z_{min} + H(Q_{min}|P) + x \end{aligned} \tag{23}$$

Moreover, the values of the dual problems converge:

$$\lim_{m\to\infty} \phi_{2m}(y_m, Z_{2m}) = \phi_{exp}(y, Z_{min}),$$

so also the value functions of the primal problem:

$$\lim_{m\to\infty} E(u_{2m}(X_0^{(2m)}(x))) = \lim_{m\to\infty} V_{2m}(x) = V_{exp}(x) = E(U_{exp}(X_0^{exp}(x))).$$

If the second assumption holds true , e.g. in a Brownian setting with deterministic coefficients, the dual problem of the 2m-th and the exponential problem have the same solution up to the constant $\mathcal{Y}_i(x)$, the density of the minimal entropy martingale measure times $\mathcal{Y}_i(x)$ for $i = 2m, exp$. The terminal values in (23) converge P almost surely and in $L^{\tilde{p}}$ for all $\tilde{p} \geq 1$.

■

Note that both assumptions imply $\mathcal{M}_e^q \neq \emptyset$. Further, under assumption 2 the terminal value of the trade-off process is bounded and so assumption 1 holds. To prove Theorem 7, we need to establish the following three steps under assumption 1 (L^1-convergence) or 2 (a.s.-convergence):

(1) a) $(Z_{2m}2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}))_m$ is uniformly integrable.
    b) $2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}})_m$ is uniformly integrable.
(2) $Z_{2m} \stackrel{L^1/a.s.}{\longrightarrow} Z_{min}$, $m \to \infty$, $(Z_{2m} := Z_T^{(\frac{2m}{2m-1})})$
(3) For every positive, real sequence $(y_m)_m$ with limit y:

$$\begin{aligned} y_m Z_{2m} I_{2m}(Z_{2m}y_m) &= y_m Z_{2m} 2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}) \\ &\stackrel{L^1/a.s}{\longrightarrow} -yZ_{min}\log(Z_{min}y) = yZ_{min}I_{exp}(Z_{min}y) \end{aligned}$$

Uniform integrability (using assumption 1) or almost sure convergence in item 3 (assumption 2), the fact that Z_{min} and Z_{2m} are strictly positive for all m, and item 2 yield $I_{2m}(Z_{2m}y_m) \stackrel{L^1/a.s.}{\longrightarrow} I_{exp}(Z_{min}y)$ for any positive real sequence (y_m) with limit y. Further, it is well-known that for a sequence $(\xi_n) \geq 0$ with $E\xi_n < \infty$ converging in probability to ξ, we have that $E\xi_n \to E\xi$ if and only if $(\xi_n)_n$ is uniformly integrable. So to prove that $\mathcal{X}_{Z_{2m}(2m)}(y)$ converges to

$\mathcal{X}_{exp}(y)$, we need that $(Z_{2m}I_{2m}(yZ_{2m}))_m$ is uniformly integrable and bounded from above. Note, $x \cdot 2m(1 - x^{\frac{1}{2m-1}})$ is bounded by two from above, see (24) below. Since $(\mathcal{X}_{2m})_m$ converge, also $(\mathcal{Y}_{2m})_m$ does and so

$$X_0^{(2m)}(x) = I_{2m}(Z_{2m}\mathcal{Y}_{2m}(x)) \overset{a.s./L^1}{\longrightarrow} I_{exp}(Z_{min}\mathcal{Y}_{exp}(x)) = X_0^{exp}(x).$$

By item 3, we have convergence of the dual functions:

$$\begin{aligned} &\phi_{2m}(\mathcal{Y}_{2m}(x), Z_{2m}) \\ &= E(-\mathcal{Y}_{2m}(x)Z_{2m}2m(1 - (Z_{2m}\mathcal{Y}_{2m}(x))^{\frac{1}{2m-1}}) - \mathcal{Y}_{2m}(x)^{\frac{2m}{2m-1}} \\ &\to E(-\mathcal{Y}_{exp}(x)Z_{min}\log(Z_{min}\mathcal{Y}_{2m}(x))) - \mathcal{Y}_{exp}(x) = \phi_{exp}(\mathcal{Y}_{exp}(x), Z_{min}) \end{aligned}$$

By duality on the 2m-th levels and in the exponential case, we have convergence of the primal value functionals:

$$\lim_{m\to\infty} V_{2m}(x) = \lim_{m\to\infty} \phi_{2m}(\mathcal{Y}_{2m}(x), Z_{2m}) = \phi_{exp}(\mathcal{Y}_{exp}(x), Z_{min}) = V_{exp}(x)$$

Finally, in the deterministic trade-off case $X_0^{(2m)}(x)$ converges to $X_0^{(\mathrm{exp})}(x)$ almost surely and in L^1. As for all $m \geq 1$ and for all $\tilde{q} > 1$ $Z_{2m} = \hat{Z}_T \in L^{\tilde{q}}$ (since $\hat{K}_T$ is deterministic), $X_0^{(2m)}(x) \in L^{\tilde{p}}$ for all $m \geq 1$ and for all $\tilde{p} \geq 1$. Hence, we find that $X_0^{(2m)}(x)$ converges in all $L^{\tilde{p}}$, $\tilde{p} \geq 1$ (this obviously also holds for $\tilde{p} \neq 2m$).

We start to prove item 1a:

PROOF. We consider the function $x \cdot 2m(1 - x^{\frac{1}{2m-1}})$. For every $\epsilon > 0$, there exists an m_0, choose $m_0 = \frac{1}{2\epsilon} + \frac{1}{2}$, such that for all $m \geq m_0$:

$$\begin{aligned} &|x \cdot 2m(1 - x^{\frac{1}{2m-1}})| \\ &\leq 2 \cdot x \int_x^1 1^{\frac{1}{2m-1}} u^{-1} du \cdot 1_{(x\in(0,1))} + 2 \cdot x \int_1^x u^{\frac{1}{2m_0-1}-1} du \cdot 1_{(x\geq 1)} \\ &\leq 2 \cdot 1 \cdot x(-\log x) 1_{(x\in(0,1))} + x(2(2m_0 - 1)(x^{\frac{1}{2m_0-1}} - 1)) 1_{(x\geq 1)} \\ &\leq 2 \cdot 0.4 + 2\epsilon^{-1} x^{\epsilon} x \end{aligned} \tag{24}$$

Note, $\epsilon = \frac{1}{2m_0-1}$ and $x(-\log x)1_{(x\in(0,1))} \leq 0.4$. In the case of assumption 2, this implies uniform integrability, since every constant sequence of a non-negative integrable random variable (in this case $(Z_{min}^{1+\epsilon})_m = (Z_{mmm}^{1+\epsilon})_m$) is uniformly integrable. Under assumption 1, by (24) it is sufficient to show that $Z_{2m}^{1+\epsilon}$ is uniformly integrable. This is established by the de la Vallé-Poussin Theorem

(VPT). As in Santacroce (2005) (proof of Theorem 1) also using a result in Kazamaki (1994), we have for a positive constant K and some $\tilde{\mu} > 1$ that

$$\sup_{1<q\leq q_0} E(|Z_T^{(q)}|^{1+\tilde{\mu}}) < K. \tag{25}$$

Next, we apply VPT to the function $G(t) = t^{1+\epsilon_2}$, where $\epsilon_2 > 0$ is still arbitrary, which obviously fulfills the assumptions in VPT. We liked to prove that $(Z_{(q)}^{1+\mu})_{q\leq q_0}$ is uniformly integrable for a $\mu > 0$. So we have to show that:

$$\sup_{q\leq q_0} E(G(|Z_q|^{1+\mu})) = \sup_{q\leq q_0} E((|Z_q|^{1+\mu})^{1+\epsilon_2}) < \infty.$$

So choose $\epsilon_2 > 0$ and $\mu > 0$ such that $\tilde{\mu} = \mu + \epsilon_2 + \mu\epsilon_2 = (1+\mu)(1+\epsilon_2) - 1$ and the assertion follows from (25) and VPT.

PROOF. *(item 1b):* Since $y_m Z_{2m} > 0$ and by the second last inequality of (24), we have:

$$\begin{aligned}
|2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}| &\leq -2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)} + 2y_m^{\epsilon}\epsilon^{-1}Z_{2m}^{\epsilon} \\
&\leq -2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)} \\
&\quad +2y_m^{\epsilon}\epsilon^{-1}(Z_{2m}1_{(Z_{2m}\geq 1)} + 1_{(Z_{2m}\in(0,1))}) \\
&\leq -2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)} + 2y_m^{\epsilon}\epsilon^{-1}(Z_{2m}+1)
\end{aligned} \tag{26}$$

We know that Z_{2m} is uniformly integrable and so also $2y_m^{\epsilon}\epsilon^{-1}(1+Z_{2m})$ (y_m converges to a real number). It remains to show that $(-2)\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}$ is uniformly integrable. We show that:

$$\log Z^{(2m)} \overset{\mathcal{H}^1}{\rightarrow} \log Z^{(min)} \tag{27}$$

without using that $Z_{2m} \overset{L^1}{\rightarrow} Z_{min}$. This yields convergence in probability of $(-2)\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}$ to $(-2)\log(yZ_{min})1_{yZ_{min}\in(0,1)}$ and L^1-integrability. Further, we know that $-2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}$ is non-negative for all m. It remains to show that

$$E(-2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}) \rightarrow E(-2\log(yZ_{min})1_{yZ_{min}\in(0,1)}). \tag{28}$$

to conclude that the sequence $-2\log(y_m Z_{2m})1_{y_m Z_{2m}\in(0,1)}$ is uniformly integrable and therefore also $2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}$. To prove (28) it suffices to show that $E(\log Z_{2m})$ converges to $E(\log Z_{min})$, since this is satisfied if and only if $E(\log(y_m Z_{2m}))$ converges to $E(\log(yZ_{min}))$ for every real positive sequence $(y_m)_m$ converging to y. Further, $(\log x_n)^- = (\log x_n)1_{x_n\in(0,1)}$ converges if and only if $(\log(x_n)1_{x_n\in[1,\infty)}) = (\log x_n)^+$ and $\log(x_n)$ converge. We already have convergence in probability and for large enough m: $\log(y_m Z_{2m})1_{y_m Z_{2m}\in[1,\infty)} \leq$

$(y+K)Z_{2m}$, where $(y+K)Z_{2m}$ is uniformly integrable. Hence the expectation of the positive part converges. $E(\log Z_{2m}) \to E(\log Z_{min})$ follows from (27). To show (27), we have by (11) and (13) that for $q = \frac{2m}{2m-1}$:

$$\begin{aligned}\log Z_{min} &= M^{Q_{min}} - \frac{1}{2}\langle M^{Q_{min}}\rangle_T = -\int_0^T \hat{\lambda}' dM_s - \tilde{m}_T - \frac{1}{2}\langle M^{Q_{min}}\rangle_T,\\ \log Z_{2m} &= -\int_0^T \hat{\lambda}' dM_s - \frac{1}{q-1}\int_0^T \frac{1}{V_s(q)} d\tilde{m}_s(q) - \frac{1}{2}\langle M^{Q_q}\rangle_T\end{aligned}$$

and $\langle M^{Q_{min}}\rangle_T = \hat{K}_T - 2\langle -\int \hat{\lambda}' dM_s, \tilde{m}_s\rangle_T + \langle \tilde{m}\rangle_T = \hat{K}_T + \langle \tilde{m}_s\rangle_T$ since $\tilde{m}$ and M are orthogonal. Similar $\langle M^{Q_q}\rangle_T = \hat{K}_T + \langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q)\rangle_T$. Finally, let $Z^{(2m)} = (E(Z_{2m}|\mathcal{F}_t))_t$ and $Z^{min} = (E(Z_{min}|\mathcal{F}_t))_t$:

$$\begin{aligned}&\|\log Z^{(2m)} - \log Z^{min}\|_{\mathcal{H}^1}\\ =&\|M^{Q_q} - \frac{1}{2}\langle M^{Q_q}\rangle - M^{Q_{min}} + \frac{1}{2}\langle M^{Q_{min}}\rangle\|_{\mathcal{H}^1}\\ =&\| - \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) + \tilde{m} + \frac{1}{2}(\langle M^{Q_{min}}\rangle - \langle M^{Q_q}\rangle)\|_{\mathcal{H}^1}\\ \le&\| - \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) + \tilde{m}\|_{\mathcal{H}^2} + \|\frac{1}{2}(\langle M^{Q_{min}}\rangle - \langle M^{Q_q}\rangle)\|_{\mathcal{H}^1} \to 0\end{aligned}$$

The first term is equal to $E^{\frac{1}{2}}(\langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) - \tilde{m}\rangle_T)$ and converges to zero for $q \downarrow 1$ by Corollary 2 in Santacroce (2005). The same corollary can be applied for the convergence of the second term. Note, by Theorem 4.5. and Proposition 4.7. in Grandits and Rheinländer (2002a) the Log-Reverse-Hölder-inequality (LRH) (for a definition see e.g. the mentioned paper) for Z_{min} is equivalent to assumption 2.1 B. So under the last assumption, by their Lemma 4.6, Condition (S) is satisfied and finally by their Lemma 2.2 $M^{Q_{min}} \in BMO(P)$. The inequalities of Kunita-Watanabe and Hölder yield

$$\begin{aligned}&\|\frac{1}{2}(\langle M^{Q_{min}}\rangle - \langle M^{Q_q}\rangle)\|_{\mathcal{H}^1} = \|\int_0^T |d(\langle M^{Q_{min}}\rangle - \langle M^{Q_q}\rangle)|\|_{L^1}\\ =&\|\int_0^T |d(\langle M^{Q_{min}} + M^{Q_q}\rangle, \langle M^{Q_{min}} - M^{Q_q}\rangle)|\|_{L^1}\\ \le&\|\sqrt{\langle M^{Q_{min}} + M^{Q_q}\rangle_T}\sqrt{\langle M^{Q_{min}} - M^{Q_q}\rangle_T}\|_{L^1}\\ \le&E^{\frac{1}{2}}\langle M^{Q_{min}} + M^{Q_q}\rangle_T E^{\frac{1}{2}}\langle M^{Q_{min}} - M^{Q_q}\rangle_T \qquad (29)\\ \le&(\sqrt{2}E^{\frac{1}{2}}\langle M^{Q_{min}} - M^{Q_q}\rangle_T\\ &+\sqrt{8}E^{\frac{1}{2}}\langle M^{Q_{min}}\rangle_T)E^{\frac{1}{2}}(\langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) - \tilde{m}\rangle_T)\\ \le&K(E^{\frac{1}{2}}(\langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) - \tilde{m}\rangle_T) + E(\langle \frac{1}{q-1}\int \frac{1}{V_s(q)} d\tilde{m}_s(q) - \tilde{m}\rangle_T))\end{aligned}$$

since $M^{Q_{min}} \in BMO(P)$. Note that we imitate here a sort of Fefferman's inequality (see Meyer (1976)) which would give the same result. It follows that $Z_{2m} \overset{P}{\rightarrow} Z_{min}$ and since $(Z_{2m})_m$ is uniformly integrable, we have $Z_{2m} \overset{L^1}{\rightarrow} Z_{min}$. Using that $\sup_t Z_t^{(2m)}$ is uniformly integrable, Doob's inequality yields convergence in $\mathcal{H}^1$ of $Z^{(q)}$ to Z^{min}.

PROOF. *(item 3):* For $x, y > 0$, we have,

$$x = \arg\max_{z\in[x,y]}(|(zI_{2m}(z))'|) = \arg\max_{z\in[x,y]}(|zI'_{2m}(z) + I_{2m}(z)|)$$

since $(zI'_{2m}(z) + I_{2m}(z))' < 0$, for $z > 0$. By an application of the mean value theorem, we have for $x < y$ and $m \geq m_0 = \frac{1}{2\epsilon} + \frac{1}{2}$:

$$\begin{aligned} |xI_{2m}(x) - yI_{2m}(y)| &= |x2m(1 - x^{\frac{1}{2m-1}}) - y2m(1 - y^{\frac{1}{2m-1}})| \\ &\leq |xI'_{2m}(x) + I_{2m}(x)||x - y| \qquad (30) \\ &= |x\frac{-2m}{2m-1}x^{\frac{1}{2m-1}-1} + 2m(1 - x^{\frac{1}{2m-1}})||x - y| \\ &\leq (2 \cdot \max\{1, |x^{\frac{1}{2m-1}}|\} + 2\epsilon^{-1}|x^\epsilon| + \frac{0.8}{x})|x - y| \end{aligned}$$

See (24) for the second last inequality. By (30), we obtain:

$$\begin{aligned} &|Z_{2m}y_m2m(1 - (Z_{2m}y_m)^{\frac{1}{2m-1}}) - (-Z_{min}y\log(Z_{min}y))| \\ \leq &|y_mZ_{2m}2m(1 - (Z_{2m}y_m)^{\frac{1}{2m-1}}) - 2myZ_{min}(1 - (Z_{min}y)^{\frac{1}{2m-1}})| \\ &+|2myZ_{min}(1 - (Z_{min}y)^{\frac{1}{2m-1}}) - (-yZ_{min}\log(Z_{min}y))|) \\ \leq &2\Big(\max\{1, (y_mZ_{2m})^{\frac{1}{2m-1}}, (yZ_{min})^{\frac{1}{2m-1}}\} \qquad (31) \\ &+\epsilon^{-1}(\max\{yZ_{min}, y_mZ_{2m}\})^\epsilon + \frac{1}{\min\{Z_{min}y, Z_{2m}y_m\}}\Big)|Z_{2m}y_m - Z_{min}y| \\ &+|2myZ_{min}(1 - (Z_{min}y)^{\frac{1}{2m-1}}) - (-yZ_{min}\log(Z_{min}y))| \overset{P}{\longrightarrow} 0 \end{aligned}$$

for any positive, real-valued sequence $(y_m)_m$ with limit y, e.g. $(\mathcal{Y}_{2m}(x))$ for fixed x. Since $Z_{2m}y_m2m(1 - (Z_{2m}y_m)^{\frac{1}{2m-1}})$ is uniformly integrable, convergence in L^1 follows.

This completes the proof.

Remark 8 *Note that from (25) the convergence of Z_{2m} also holds in an $L^{1+\epsilon}-$space for an $\epsilon > 0$ and $Z_{min} \in L^{1+\epsilon}$. This follows directly from uniform integrability and the convergence in probability.*

5 Convergence to the Optimal Portfolio for an Exponential Utility Function

We turn to the question of convergence of the corresponding portfolios, namely whether the optimal portfolios N^{2m} of the $2m$-problems converge to the optimal portfolio N^{exp} of the exponential problem. Here we will restrict our considerations to the case where assumption 2 of Theorem 7 holds ($\hat{K}_T = \langle -\int \hat{\lambda}' dM \rangle_T$ is deterministic), for some ideas on a more general setting see remark 11.
The basic idea used to derive convergence of the optimal controls/portfolios consists in considering X_0^{2m}, X_0^{exp} as the terminal values of a BSDE describing the price of the terminal values. The two components of the solutions of these BSDEs are derived and the second parts of the solutions corresponding to the optimal portfolios are shown to converge. Finally, we consider the case of a Brownian market with deterministic coefficients $\mu(t), \sigma(t)$, and $r = 0$.
We start searching for a portfolio q^{2m} that reaches X_0^{2m} by solving the corresponding pricing equation and show that these portfolios converge to a price process with terminal value $X_0^{(exp)}$. The pricing equation for the claim X_0^{2m} is of the following form:

$$\begin{aligned} dp_t^{(2m)} &= (q_t^{(2m)})' d\langle M \rangle_t \hat{\lambda}_t + (q_t^{(2m)})' dM_t + dL_t^{(2m)}, \\ p^{(2m)}(T) &= X_0^{(2m)}(x) \end{aligned} \tag{32}$$

where $L^{(2m)}$ is the orthogonal term appearing in the Föllmer-Schweizer decomposition. q^{2m} represents the portfolio. Since X_0^{2m} is attainable, $L^{(2m)}$ vanishes and we have that (32) is uniquely solvable. The above BSDE is linear so we can look for a portfolio by considering

$$p_t^{(2m)} := \hat{Z}_t^{-1} E(\hat{Z}_T X_0^{(2m)}(x) | \mathcal{F}_t) \tag{33}$$

as a possible candidate. Itô's formula and a coefficient comparison then yield that this process p^{2m} is in fact equal to the optimal price process $Y^{(2m)}$ that reaches $X_0^{(2m)}$.

Before starting these calculations and introducing an example, again remember that in the present situation the minimal martingale measure (respectively its density) coincides with the q-optimal martingale measure for all q. With $Z_{2m} = \hat{Z}_T$ for all m, we find from (22)

$$X_0^{(2m)}(x) = 2m \left(1 - \hat{Z}_T^{\frac{1}{2m-1}} (1 - \frac{x}{2m}) E^{-1} \left(\hat{Z}_T^{\frac{2m}{2m-1}} \right) \right).$$

By Novikov's condition $\hat{Z}_q = \mathcal{E}_T(-\int q\hat{\lambda}' dM)$ is a martingale for every $q \in \mathbb{R}$

and therefore $E(\hat{Z}_q) = 1$. It follows that $\hat{Z} \in L^q(P)$ for every arbitrary $q \geq 1$, because

$$\begin{aligned}\hat{Z}^q =& \exp\{-\int_0^T q\hat{\lambda}'_s dM_s - \frac{1}{2q}\int_0^T q^2\hat{\lambda}'_s d\langle M\rangle s\hat{\lambda}_s\} \\ =& \exp\{-\int_0^T q\hat{\lambda}'_s dM_s - \frac{1}{2}\int_0^T q^2\hat{\lambda}'_s d\langle M\rangle_s\hat{\lambda}_s + \frac{q-1}{2q}\int_0^T q^2\hat{\lambda}'_s d\langle M\rangle_s\hat{\lambda}_s\}\end{aligned} \tag{34}$$

and since $\langle \int \hat{\lambda}' dM\rangle$ is deterministic:

$$\begin{aligned}E(\hat{Z}^q) =& E(\exp\{-\int_0^T q\hat{\lambda}'_s dM_s - \frac{1}{2}\int_0^T q^2\hat{\lambda}'_s d\langle M\rangle_s\hat{\lambda}_s \\ &+ \frac{q-1}{2q}\int_0^T q^2\hat{\lambda}'_s d\langle M\rangle_s\hat{\lambda}_s\}) \\ =& \exp\{\frac{q^2(q-1)}{2q}\int_0^T \hat{\lambda}'_s d\langle M\rangle_s\hat{\lambda}_s\} < \infty\end{aligned} \tag{35}$$

By plugging (34) and (35) into (33) and applying Itô's formula we find that $(p_t^{(2m)}, q_t^{(2m)})$ uniquely solves (32), where

$$\begin{aligned}p_t^{(2m)} :=& \hat{Z}_t^{-1}E(\hat{Z}_T X_0^{(2m)}(x)|\mathcal{F}_t) \\ =& 2m(1 - \exp(-\frac{1}{2m-1}\int_0^t \hat{\lambda}'_s dM_s - \frac{1}{2(2m-1)^2}\int_0^t \hat{\lambda}'_s d\langle M\rangle_s\lambda'_s) \\ &\times(\exp(\frac{-1}{(2m-1)}\int_0^t \hat{\lambda}'_s d\langle M\rangle_s\lambda'_s)(1 - \frac{x}{2m}))) \\ =:& 2m(1 - z_t^{(2m)}\beta_t^{(2m)}(1 - \frac{x}{2m})) \text{ (optimal 2m-th wealth process)}\end{aligned}$$

and

$$q_t^{(2m)} = \frac{(2m-x)}{(2m-1)}\hat{\lambda}_t z_t^{(2m)}\beta_t^{(2m)} =: (N^{(2m)}(t))' \text{ (portfolio process)}$$

With this we have found the optimal portfolios for the $2m$-problems.
Next we turn to the convergence of the solutions of the $2m$-level BSDEs to the BSDE of the exp-problem: Since $\int \hat{\lambda}' dM$ is continuous, we have that

$$\frac{2m-x}{2m-1}\hat{\lambda} z^{(2m)}\beta^{(2m)} \to \hat{\lambda}, \quad P\text{.a.s} \tag{36}$$

uniformly in t. Since $\langle \int \hat{\lambda} dM\rangle_T$ is deterministic, we further show for all $\tilde{p} > 1$:

$$E(\sup_t |p_t^{(2m)} - p_t|^{\tilde{p}}] \to 0, m \to \infty, \tag{37}$$

where $p_t := x + \int_0^t \hat{\lambda}' dS$. Finally, we already know that $p_T^{(2m)} \overset{L^1}{\to} X_0^{(exp)}(x)$, which yields

$$p_T = X_0^{(exp)}(x).$$

Hence the optimal portfolio that reaches $X_0^{(exp)}$ is equal to:

$$N^{(exp)} = \hat{\lambda}' \in \mathcal{A}^{exp}.$$

where $\mathcal{A}^{exp}$ is defined in Remark 2. After establishing (36), (37) follows from the dominated convergence theorem, and $\hat{\lambda}' \in \mathcal{A}^{exp}$, from our assumption. We thus get the following theorem:

Theorem 9 *If $\langle \int \hat{\lambda}' dM \rangle_T$ is deterministic, then $(\hat{\lambda}', 0) \in \mathcal{A}^{exp} \times \mathcal{K}$ is the optimal portfolio of the problem*

$$V_{exp}(x) = \max_{(N,C) \in \mathcal{A}^{exp} \times \mathcal{K}} E(1 - e^{-(x + \int_0^T NdS - C_T)}), \tag{38}$$

where $\mathcal{K}$ is an arbitrary class of right-continuous increasing processes. Further,

$$E(\sup_t |p_t^{(2m)} - (x + \int^t \hat{\lambda}' dS)|^{\tilde{p}}) \to 0, m \to \infty, \tilde{p} \geq 1 \tag{39}$$

where $p_\cdot^{(2m)}$ is the optimal wealth process of

$$V(x)_{2m} \equiv \sup_{(N,C) \in \mathcal{A}^p \times \mathcal{K}^p} E[-(1 - \frac{x + \int_0^T NdS - C_T}{2m})^{2m}],\ x \in \mathbb{R} \tag{40}$$

Finally, we establish the equality $X_0^{exp} = x + \int_0^T \hat{\lambda}' dS$.

Before proving the last Theorem, we apply these results to a Brownian case:

Example 10 *We consider an n-dimensional stock:*

$$S_t = S_0 + \int_0^t \mathbf{S}_s \mu(s) ds + \int_0^t \mathbf{S}_s \sigma(s) dW_s, \tag{41}$$

where W is a $d-$dimensional Brownian Motion, $n \leq d$, σ a deterministic $n \times d-$ matrix, and $\mathbf{S} = \mathrm{diag}(S^{(1)}, .., S^{(n)})$. We have $\hat{\lambda} = \mu'(\sigma\sigma')^{-1}\mathbf{S}^{-1}$ and so $\hat{Z}$ is of the form:

$$\hat{Z} = \exp\{-\int_0^t \bar{\theta}_s' dW_s - \frac{1}{2}\int_0^t \|\bar{\theta}_s\|^2 ds\},$$

where $\bar{\theta} = \sigma'(\sigma\sigma')^{-1}\mu$. All assumptions in Frittelli's Theorems are satisfied by the minimal martingale measure (note that all coefficients are bounded), see El Karoui and Quenez (1995). With $\hat{\lambda}_t = \mathbf{S}_t^{-1}(\sigma_t\sigma_t')^{-1}\mu_t$, (32) is uniquely *solvable by* (p_t^{2m}, q_t^{2m}). *On the other hand*

$$z_t^{(2m)}\beta_t^{(2m)} = (1 - \frac{p_t^{(2m)}}{2m})\frac{2m}{(2m-x)}. \tag{42}$$

Hence, with

$$\tilde{q}_t^{(2m)} = \frac{2m}{2m-1}\bar{\theta}_t(1 - \frac{p_t^{(2m)}}{2m}),$$

$(p^{(2m)}, \tilde{q}^{(2m)})$ *is the optimal solution of*

$$dp_t^{(2m)} = \bar{\theta}_t' \tilde{q}_t^{(2m)} dt + (\tilde{q}_t^{(2m)})' dW_t, \quad p^{(2m)}(T) = X_0^{(2m)}(x)$$

where $(\tilde{q}_t^{(2m)})' = (q_t^{(2m)})' \mathbf{S}_t \sigma_t$. *So*

$$\pi^{(2m)} = \frac{2m}{2m-1}(\sigma\sigma')^{-1}\sigma\bar{\theta}(1 - \frac{Y_\cdot^{(2m)}}{2m}) = \mathbf{S}q^{2m} \to \mathbf{S}\hat{\lambda} = (\sigma\sigma')^{-1}\sigma\bar{\theta} =: \pi^{(exp)},$$

where $\pi^{(2m)}$ *the amount invested in the stocks* $\mathbf{S}_t$.
Finally summarizing the above results we have the following "commuting" diagram where $\cong$ *should be read as "corresponds to in the above explained sense":*

$$\begin{array}{ccccc}
\mathcal{Y}_{2m}(x)Z_{2m} & \longrightarrow & \mathcal{Y}_{exp}(x)Z_{min} & & \textit{(convergence of dual solutions)} \\
\cong & \circlearrowleft & \cong & & \\
X_0^{(2m)}(x) & \longrightarrow & X_0^{(exp)}(x) & & \textit{(convergence of terminal wealths)} \\
\cong & \circlearrowleft & \cong & & \\
\pi^{(2m)} & \longrightarrow & \pi^{exp} & & \textit{(convergence of portfolios)} \\
\cong & \circlearrowleft & \cong & & \\
V_{2m}(x) & \longrightarrow & V_{exp}(x) & & \textit{(convergence of value functions)}
\end{array}$$

■

PROOF. (of Theorem 9) We start by establishing (36). For (36), we just have to consider $z^{(2m)}$ and $\beta^{(2m)}$. Since $\int_0^\cdot \hat{\lambda}' dM$ is continuous, we have that for arbitrary ω

$$\left|\frac{1}{2m-1}\sup_t(-\int_0^t \hat{\lambda}' dM)\right| \leq K(\omega)\frac{1}{2m-1} \to 0.$$

Similar for $\langle \int \hat{\lambda}' dM \rangle$. And so the power of β^{2m} and z^{2m} converge to zero. By defining $g^{2m} := \frac{2m-x}{2m-1} z^{(2m)}\beta^{(2m)}$, we get by Burkholder-Davis-Gundy-inequality

$$
\begin{aligned}
&E(\sup_t |\int_0^t (q_s^{2m} - \hat{\lambda}_s)' dS_s|^{\tilde{p}}) \\
&\leq K_{\tilde{p}} E(\sup_t |\int_0^t (g_s^{2m} - 1)\hat{\lambda}_s' d\langle M\rangle_s \hat{\lambda}_s|^{\tilde{p}}) + \sup_t |\int_0^t (g_s^{2m} - 1)\hat{\lambda}_s' dM_s|^{\tilde{p}}) \\
&\leq K_{\tilde{p}} E(\sup_t |g_t^{2m} - 1| \int_0^t |\hat{\lambda}_s' d\langle M\rangle_s \hat{\lambda}_s|^{\tilde{p}}) + E(\sup_t |\int_0^t (g_s^{2m} - 1)^2 \hat{\lambda}_s' d\langle M\rangle_s \hat{\lambda}_s|^{\frac{\tilde{p}}{2}}) \\
&\leq K_{\tilde{p}} E(\sup_t |g_t^{2m} - 1|^{\tilde{p}})(\langle \int_0^\cdot \hat{\lambda}_s' dM_s\rangle_T^{\tilde{p}} + \langle \int_0^\cdot \hat{\lambda}_s' dM_s\rangle_T^{\frac{\tilde{p}}{2}}) \\
&\leq \tilde{K}_{\tilde{p}} \cdot E(\sup_t |g_t^{2m} - 1|^{\tilde{p}})
\end{aligned}
$$

By the dominated convergence theorem and (36), it remains to show that $\max_t |g_t^{2m}|^{\tilde{p}}$ is dominated by an integrable random variable:

$$
\begin{aligned}
&\exp(-\frac{1}{2m-1} \int_0^t \hat{\lambda}_s' dM_s) \\
&\leq 1 + \exp(-\int_0^t \hat{\lambda}_s' dM_s) \exp(-\frac{1}{2}\langle \int_0^\cdot \hat{\lambda}_s' dM_s\rangle_t) \exp(\frac{1}{2}\langle \int_0^\cdot \hat{\lambda}_s' dM_s\rangle_t) \\
&\leq 1 + K z_t^{(2)}
\end{aligned}
$$

By Doob's inequality and (34), we have for all $\tilde{p} > 1$:

$$
E \max |g_t^{(2m)}|^{\tilde{p}} \leq K_1(1 + KE \max_t |z_t^{(2)}|^{\tilde{p}}) = K_1(1 + KE|\hat{Z}_T|^{\tilde{p}}) < \infty
$$

Finally, $\hat{\lambda}' \in \mathcal{A}^{exp}$, i.e. $\|\hat{\lambda}'\|_{L^{\tilde{p}}(M)} = \|(\int_0^T \hat{\lambda}' d\langle M\rangle_t \hat{\lambda})^{\frac{1}{2}}\|_{L^{\tilde{p}}} < \infty$ and

$$
\|\hat{\lambda}'\|_{L^{\tilde{p}}(A)} = \|\int_0^T |N||dA_t|\|_{L^{\tilde{p}}(A)} = \|\int_0^T |\hat{\lambda}_t' d\langle M\rangle_t \hat{\lambda}_t|\|_{L^{\tilde{p}}(A)} < \infty
$$

holds since $\int_0^T \hat{\lambda}' d\langle M\rangle_t \hat{\lambda}$ is deterministic and $\int_0^T \hat{\lambda}' dS = X_0^{exp} \in \mathcal{O}^{\tilde{p}}$.

Remark 11 *(i) Obviously, our estimations in the proof of Theorem 9 heavily rely on the assumption that $\hat{K}_T$ is deterministic. Typically, if this assumption is not made, $Z^{(2m)}$ are all different and do not coincide with $\hat{Z}$, see Pham et al. (1998). Nevertheless, in the general setting of the first sections and under some other strong conditions the above method also works. We shortly sketch the idea without giving the technical proofs. Since $X_0^{(2m)}(x) = I_{2m}(\mathcal{Y}_{2m}(x) Z_T^{(2m)})$ is attainable, the solution of (32) is easily guessed to be*

$$
p_t^{(2m)} = (Z_t^{(2m)})^{-1} E(Z_T^{(2m)} X_0^{(2m)}(x)|\mathcal{F}_t) \tag{43}
$$

A lengthy and tedious calculation gives the following results, for $q = \frac{2m}{2m-1}$:

$$
p_t^{(2m)} = 2m - 2m(Z_t^{(2m)})^{\frac{1}{2m-1}} E((\mathcal{E}_{tT}(M^{Q_q}))^{\frac{2m}{2m-1}}|\mathcal{F}_t) E^{-1}((Z_T^{(2m)})^{\frac{2m}{2m-1}}) \tag{44}
$$

Now apply Theorem 5 to represent the density process $Z^{(2m)}$ as the exponential of

$$M_t^{Q_q} = -\int_0^t \hat{\lambda}_s' dM_s - \frac{1}{q-1}\int_0^t \frac{1}{V_s(q)} d\tilde{m}_s(q) \tag{45}$$

to find:

$$Z_t^{(2m)} = \mathcal{E}_t(-\int \hat{\lambda}' dM - \frac{1}{(q-1)}\int \frac{1}{V_s(q)} d\tilde{m}_s(q)),\ q = \frac{2m}{(2m-1)} \tag{46}$$

Next separate out of $(Z^{(2m)})^{\frac{1}{2m-1}}$ the exponential martingales $\tilde{z}_t^{(2m)}$ by making use of a Novikov condition to find a representation of $p_t^{(2m)}$ similar to the above representation. Formally, $q_t^{(2m)}$ turns out to have the same form as above. Finally, prove that

$$q_t^{(2m)} = \frac{(2m-x)}{(2m-1)} \hat{\lambda}_t \tilde{z}_t^{(2m)} \tilde{\beta}_t^{(2m)} =: N^{(2m)}(t)$$

is the portfolio process, and the convergence

$$\frac{2m-x}{2m-1} \hat{\lambda} \tilde{z}^{(2m)} \tilde{\beta}^{(2m)} \to \hat{\lambda}, \quad P\text{.a.s} \tag{47}$$

When deriving these calculations one must carefully treat the orthogonal terms in the GKW decomposition and keep in mind that the terminal values are all attainable. This gives the desired result in the framework of this article. A sufficient (very strong) set of assumptions in the Brownian setting above is the following: μ is bounded for all t and ω and σ is bounded away from zero and bounded above (Reverse Hölder inequality is satisfied). The result turns out to be formally identical with the result in the very restricted setting of the above example. These results will be extended by making use of techniques different from those in this article. E.g. using a (here: localized version of a) generalization of the monotone stability Proposition 2.4 of Kobylanski (2000) (also see Briand et al. (2003)) to derive the convergence of the portfolios from the convergence of the terminal values of a family of BSDEs. A major difficulty in the general setting is to overcome a boundedness condition like $X_0^{\exp} \in L^\infty$, see e.g. Briand and Hu (2005).
(ii) By making use of the standard change of numéraire techniques it is easily seen that for $r \neq 0$ the above result holds with μ replaced by $\mu - r \cdot \mathbf{1}$. By approximating $\exp(-\alpha(Y_T - \xi))$, $Y_T := x + \tilde{Y}_T$ by the sequence

$$(1 + \frac{\alpha\xi}{2m})^{2m}(1 - \frac{\alpha Y_T}{2m})^{2m}$$

we find for the optimal portfolio of the $2m$-problem from a BSDE similar to (32) with terminal value $X_0^{(2m)} - \xi$ that

$$\pi^{(2m)} = \frac{2m}{\alpha(2m-1)} \sigma'(\sigma\sigma')^{-1}\bar{\theta}'(1 + \frac{p_\xi^{(2m)}}{2m} - \frac{Y_\cdot^{(2m)}}{2m}) - \sigma'(\sigma\sigma')^{-1} q_\xi^{(2m)}, \tag{48}$$

where $(p_\xi^{(2m)}, q_\xi^{(2m)}) = (p_\xi, q_\xi)$ *is the solution of the usual BSDE for hedging* ξ*, for simplicity a* L^∞*-random variable. The optimal control for the* $\exp -$*hedging problem thus turns out to be*

$$\pi^{\exp} = \frac{\sigma'(\sigma\sigma')^{-1}\bar{\theta}'}{\alpha} - \sigma'(\sigma\sigma')^{-1}q_\xi. \tag{49}$$

This result should be compared to the mean variance hedging result e.g. in Kohlmann and Zhou (2000): The pure mean variance hedging borrows money to hedge the claim and invests the difference between the price of the claim and the actual wealth according to a sort of Merton portfolio. Here we have a similar behavior, but an extra term appears which tries to drive the wealth higher than the claim. When looking at the structure of the functional $1 - e^{-\alpha(x+\check{Y}_T-\xi)}$ *this obviously makes perfect sense. So the exponential hedging yields a similar disadvantage as in the mean variance case: In the latter case overshooting the claim is punished, in the case under consideration here, overshooting is rewarded (see e.g. the discussion in Cont and Tankov (2006)).*

6 Conclusion

The paper provides a new and complete framework to solve the dynamic utility maximization problem for an exponential utility function via an approximation approach.
We consider control problems for the sequence of functions $-(1 - \frac{x}{2m})^{2m}$ (2m-th problem), which is rather interesting itself as a modification of isoelatic control problem, see e.g. Bürkel (2005). We start giving a solution method to solve a general dynamic utility maximization problem with not necessarily increasing, concave utility functions. In a first step, we transform the dynamic problem into a static problem via a hedging argument. For increasing utility functions this was already proven in Delbaen and Schachermayer (1996b). An extension to not everywhere increasing utility functions in an L^p setting is now given in this paper for the first time. We further present a simple method to solve the static problem applying a duality approach in section 3. Using this method, we can easily derive the optimal terminal value of the 2m-th and the exponential problem. Theorem 7 presents our main result on the relation between various kinds of optimal martingale measures, the 2m-th, and the exponential problem. Under some very weak assumption in a general semimartingale model, we can prove the convergence of the terminal values and the value functions of the 2m-th to the exponential problem.
Section 5 establishes a portfolio for the 2m-th problem in a setting with a deterministic terminal value of the trade-off process via a BSDE-approach. The above convergence result then yields strong convergence of the portfolios and gives an explicit portfolio for the exponential problem.

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