ON CONVERGENCE TO THE EXPONENTIAL UTILITY PROBLEM WITH JUMPS

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Abstract. We derive an explicit portfolio for the exponential utility maximization problem via an approximation approach for exponential Lévy processes (mainly discussing $-e^{-x}$ (exponential problem) and $-(1-\frac{x}{2m})^{2m}$ (2m-th problem)). A result by Jeanblanc et al.(2006) is applied: The convergence of q-optimal martingale measures to the minimal entropy martingale measure. Except for conditions on the existence of the q-optimal measures, we replace technical assumptions by minor integrability conditions. We obtain convergence of the portfolios of the 2mth to the exponential problem. The influence of jump intensity and jump size distribution upon the portfolio, in comparison to the continuous case, is discussed.

1. Introduction

Utility maximization problems are topics frequently investigated in mathematical finance. The aim is to find an optimal investment in n stocks and one riskless asset. An investment is optimal if it maximizes a certain utility function at a fixed time point T. The existence of such optimal portfolios is shown e.g. in Kramkov and Schachermayer [14]/Schachermayer [19] for general classes of utility functions or Delbaen et al. [3] for exponential utility functions. Moreover, an approach applying the theory of backward stochastic differential equations (BSDE) can be found in Rouge and El Karoui [16] and Hu et al. [8]. All approaches only include locally bounded stock processes, if the domain of the utility function is equal to the real line. That excludes several types of Lévy processes. Becherer

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[1] proves results, when BSDEs are driven by random measures. This enables him to solve dynamic optimization problems with exponential utility and non-continuous underlying filtration.

In a Lévy setting, KALLSEN [12] finds a portfolio for the exponential utility function and proves its optimality. In the same reference, analogous results are shown for isoelastic utility functions, but not with parameters $p > 1$ ($\|x\|^p$). A connection of both problems is not drawn. We therefore discuss transformations of isoelastic and exponential utility functions and its connection in a Lévy setting. By extending an approximation argument suggested in KOHLMANN AND NIETHAMMER [13] to the discontinuous case, we directly construct explicit portfolios for exponential and isoelastic utility functions in the case of exponential Lévy processes. A connection between both problems is automatically involved in the approximation argument.

In detail, KOHLMANN AND NIETHAMMER [13] derive a method to solve dynamic expected utility maximization problems with possibly not everywhere increasing utility functions in a continuous semimartingale/L^p-setting. We show that the method can be partly applied, if semimartingales are not continuous. To obtain a solution, the dynamic problem is reduced to a static problem over $\mathcal{F}_T$-measurable random variables satisfying a budget constraint: The expected value under certain pricing measures must not be greater than the invested initial wealth. The static optimization problem is than solved by finding an optimal pricing measure [the optimal terminal value of our wealth process is a function of the density of the optimal measure]. In particular, we find optimal terminal values of our wealth for utility functions of type $-e^{-\alpha x}$ (exponential problem) and $-(1 - \frac{\alpha x}{2m})^{2m}$ (2m-th problem), $\alpha > 0$. However, the set of pricing measures has to be large enough such that the dynamic problem can actually be reduced. For some processes, this can only be achieved if we consider signed measures. As signed pricing measures or better signed martingale measures involve no natural interpretation, we restrict our considerations to cases where optimal measures are equivalent martingale measures. This restricts the study of isoelastic functions somehow, whereas in the exponential case, signed measures do not have to be considered at all. The minimal entropy martingale measure (MEMM) is optimal, provided it actually exists.

In our Lévy setting, FUJIWARA AND MIYAHARA [6] provide an explicit representation of the MEMM under a tractable set of assumptions, whereas ESCHE AND SCHWEIZER [4] give an extension to the multivariate case. It is easy to show that all examples given in [6] satisfy a slightly stronger condition implying that the density of MEMM is in $L^{1+\tilde{\epsilon}}$. The dual solution of the 2m-th problem is the q-optimal martingale measure (qMMM) for $q = 2m/(2m-1)$. A sufficient condition for the existence of (equivalent) qMMM as well as an explicit form of the density process is provided by JEANBLANC ET AL. [10]. Furthermore, they prove convergence in entropy of the qMMMs to MEMM, unfortunately under quite technical assumptions. We replace them by an integrability condition, which is satisfied for all processes with jumps bounded from above. This includes a modified variance gamma or a normal inverse gaussian Lévy process with jumps bounded from above. Under the mentioned integrability condition, we prove uniform integrability of $(((Z_T^{(q)})^{1+\tilde{\epsilon}})_q)$, where $Z_T^{(q)}$ denotes the density of qMMM. Convergence of the densities in $L^{1+\tilde{\epsilon}}$ follows.

Convergence of q-optimal measures to the minimal entropy measure then implies L^1-convergence of the terminal values in a Lévy setting. Afterwards, we construct optimal portfolios for the 2mth and the exponential problem. We further establish convergence of the portfolios and the corresponding wealth processes uniform in probability. Moreover, convergence of the optimal wealth processes in supremums norm and convergence of the terminal values in L^r, $r \geq 1$ follows under the above integrability condition. Finally, the influence of jump intensity and jump size distribution upon the portfolio, in comparison to the continuous case, is discussed.

The paper is organized as follows. We start introducing the market model and explain the difference to the setting in [13]. The third section solves the static utility problems (2mth and exponential problem). The forth section recalls and proves some necessary results on martingale measures and its connections. Examples for regularity conditions are given. Whereas section 5 proceeds with the convergence of the static problem and the optimal portfolios. We close the paper by discussing the influence of the jump component on the portfolio.

2. The Market Model and Problem Formulation

2.1. **The Market Model.** We introduce the usual semimartingale setting and show then how exponential Lévy processes fit into this structure. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T \in (0, \infty)$ a time horizon, and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ a filtration satisfying the usual conditions, i.e. right-continuity and completeness. All expectations and spaces without a subscript are defined with respect to the measure P. K denotes a generic positive constant. L^p is always meant with respect to Ω, $\mathcal{F}_T$, and P: $L^p := L^p(\Omega, \mathcal{F}_T, P)$. Moreover, all considered semimartingales S admit a Doob- Méyer decomposition $S = S_0 + M + A$ into a local martingale M and a predictable process A of bounded variation.

In contrary to [13], our stock price process is not supposed to be continuous, but assumed to have the following form:

$$S_t = S_0 e^{\check{X}_t},\ t \in [0, T],\ S_0 \in \mathbb{R}^n \tag{1}$$

and $\check{X}$ a $\mathbb{R}^n$-valued Lévy process on $(\Omega, \mathcal{F}, P)$. The filtration $\mathbb{F}$ is supposed to be the completion of the filtration induced by $\check{X}$. By the Lévy-Itô- decomposition, (see e.g. Cont and Tankov [2]), $\check{X}$ has the following form:

$$\check{X}_t = bt + \sigma W_t + \int_0^t \int_{\|x\|>1} xN(dx, ds) + \int_0^t \int_{\|x\|\leq 1} x\tilde{N}(dx, ds) \tag{2}$$

where $\tilde{N}(dx,dt) = N(dx,dt) - \nu(dx)dt$. ν is the corresponding Lévy-measure, N a Poisson random measure with intensity measure $\nu(dx)dt$, and W a Brownian motion.

The riskless asset has a constant interest rate r.

Assumption 2.1. *$\mathcal{M}_e$ denotes the set of all probability measures Q equivalent to P such that $\tilde{S} = S_t e^{rt}$ is a Q-local martingale. It is assumed to be nonempty.*

If $\mathcal{M}_e$ is a singleton, we call the market complete, otherwise incomplete. Since our risky asset S contains jumps, the market is no longer complete. For simplicity, we only discuss the case $r = 0$. That is no major restriction, we just have to change the drift term of $\check{X}$, i.e. $B_t = B_0 e^{rt}$ serves as a numéraire. Note further, if there are no jumps, the above model corresponds to the Brownian model by setting $\mu = b - \frac{1}{2}\sigma_2$, $dS = S\mu dt + S\sigma dW$, where $\sigma_2^{(i)} = \sum_{j=1}^{d} \sigma_{ij}^2$.

Next, we write the exponential Lévy process as a special semimartingale:

$$d\mathbf{S}_t \quad = \quad \mathbf{S}_{t-} d\tilde{L}_t \tag{3}$$

with

$$\begin{aligned} d\tilde{L}_t \quad = \quad & -\beta dt + \int_{\|x\|\leq 1} (e^x - \mathbf{1} - x)\nu(dx)dt + \sigma dW_t \\ & + \int_{\|x\|>1} (e^x - \mathbf{1}) N(dx,dt) + \int_{\|x\|\leq 1} (e^x - \mathbf{1})\ \tilde{N}(dx,dt) \end{aligned} \tag{4}$$

where $\beta = -(b + \frac{1}{2}\sigma_2 - r\mathbf{1}) := -(\mu - r\mathbf{1})$ and $\mathbf{S} = diag(S^{(1)}, ..., S^{(n)})$. $\mathbf{1}$ denotes an $n \times 1$ vector of ones and e^x should be seen componentwise, i.e. $e^x = (e^{x_1}, ..., e^{x_n})'$. Note, in [10] the authors consider $S = \mathcal{E}(\tilde{L})$, where $\tilde{\nu}$ is the Lévy measure of $\tilde{L}$: $\int_{\mathbb{R}^n} 1_{\{e^x - \mathbf{1} \in A\}} \nu(dx) = \tilde{\nu}(A)$, see remark 2.3 in [10]. If $\int \mathbf{1}'(e^x - \mathbf{1})(e^x - \mathbf{1})'\mathbf{1}\nu(dx) < \infty$, ($\int \|(e^x - \mathbf{1})\|^2 \nu(dx) < \infty$ is sufficient) the Structure Condition is satisfied, i.e. : There exists a local martingale M and a predictable process $\hat{\lambda}$ with $\langle \int \hat{\lambda}' dM \rangle_T < \infty$ such that $S = S_0 + M + A$, where $A = \int d\langle M \rangle \hat{\lambda}$:

$$dM_t = \mathbf{S}_{t-}(\sigma dW_t + \int (e^x - \mathbf{1})\tilde{N}(dx,dt))$$

$$dA_t = \mathbf{S}_{t-}(-\beta + \int (e^x - \mathbf{1} - x 1_{\|x\|\leq 1})\nu(dx))dt$$

and so

$$d\langle M \rangle = \mathbf{S}_{t-}(\sigma\sigma' + \int (e^x - \mathbf{1})(e^x - \mathbf{1})'\nu(dx))\mathbf{S}_{t-} dt$$

thus $\hat{\lambda} = \mathbf{S}_{t-}^{-1} CD =: \mathbf{S}_{t-}^{-1}(-\mu_*)$ where

$$\begin{aligned} C &= \left(\sigma\sigma' + \int (e^x - \mathbf{1})(e^x - \mathbf{1})'\nu(dx)\right)^{-1} \\ D &= (-\beta + \int (e^x - \mathbf{1} - x\mathbf{1}_{\|x\|\leq 1})\nu(dx)). \end{aligned}$$

Moreover,

$$\int_0^t \hat{\lambda}'_u dM_u = -\mu'_*(\sigma W_t + \int_0^t \int (e^x - \mathbf{1})\tilde{N}(dx, du))) \tag{5}$$

and finally

$$\hat{K}_t := \langle \int_0^\cdot \hat{\lambda}'_u dM_u \rangle_t = \mu'_*(\sigma\sigma' t + t\int (e^x - \mathbf{1})(e^x - \mathbf{1})'\mu_*\nu(dx))) < \infty.$$

Hence, $\hat{K}$, the so called trade off process, is deterministic, if

Assumption 2.2. $\int \|(e^x - \mathbf{1})\|^2 \nu(dx) < \infty$

Clearly, in a context of financial data, this constraint should be intuitively satisfied. We give some examples:

Example 2.1. *For $\|x\| \leq 1$, assumption 2.2 is always satisfied, because $(e^{x_i} - 1)^2$ is of the same order as x_i^2, which is integrable by the Lévy-Itô-decomposition. Clearly, if x_i is bounded from above, we are done. Otherwise we have to ensure the integrability of e^{2x_i}. That is e.g. satisfied if the jump component is driven by a compound Poisson process with a Gaussian jump size distribution. We can also switch to the Lévy measure $\tilde{\nu}$ and set $\nu(G) = \int 1_{\log(y+\mathbf{1})\in G}\tilde{\nu}(dy)$, i.e. defining the jump size distribution of $y = e^x - \mathbf{1}$. All distributions with finite second moments are suitable.*

■

2.2. **Trading Strategies and Optimization Problem.** We continue with a suitable class of trading strategies. A self-financing strategy $(\tilde{x}, \vartheta)$ is given by the initial wealth $\tilde{x}$ and the number of shares $\vartheta = (\vartheta^1, ..., \vartheta^n)$ of stocks held at time $t \in [0, T]$. We require that our strategies are predictable and satisfy an integrability condition:

Definition 2.1. *The set of L^p-trading or p-integrable strategies is defined as follows:*

$$\mathcal{A}^p := L^p(M) \cap L^p(A)$$

where

$$L^p(M) = \{\vartheta \in \mathbb{P}^n | \ \|\vartheta\|_{L^p(M)} < \infty\}, \ \ L^p(A) = \{\vartheta \in \mathbb{P}^n | \ \|\vartheta\|_{L^p(A)} < \infty\}$$

with $\|\vartheta\|_{L^p(M)} := \|(\int_0^T \vartheta d[M]_t \vartheta')^{\frac{1}{2}}\|_{L^p}$, $\|\vartheta\|_{L^p(A)} := \|\int_0^T |\vartheta dA_t|\|_{L^p}$ and $\mathbb{P}^n$ is the set of all predictable $\mathbb{R}^n$-valued processes. In the exponential case, we consider

$$\mathcal{A}^{exp} = \{\vartheta \in \bigcap_{p>1} \mathcal{A}^p : Ee^{-\alpha\int_0^T \vartheta dS} < \infty\}.$$

Note, the exponential utility function is defined as $e^{\alpha x}$, $\alpha > 0$. Self-financing strategies in the above classes plus a consumption process (ϑ, C) are then sufficient to define a class of wealth processes:

$$\mathcal{W}_C(\tilde{x}) = \{Y | Y_t = \tilde{x} + \int_0^t \vartheta dS - C_t, \vartheta \in \mathcal{A},\ C \in \mathcal{K}^p\}$$

where $\mathcal{K}^p$ the class of increasing right-continuous processes with $C_T \in L^p$ and $\mathcal{A} = \mathcal{A}^p$ resp. $\mathcal{A}^{exp}$. Over these classes of wealth processes, the following dynamic optimization problem is considered:

$$V(x) \equiv \sup_{Y \in \mathcal{W}_C(\tilde{x})} E[U(Y_T)],\ \tilde{x} \in \mathbb{R} \tag{6}$$

where U is a concave, not necessarily increasing function.

(6) is closely connected to a set of martingale measures. We assume that these sets are non empty. Conditions in a Lévy setting are given later. For $\mathcal{A}^p$, $p \in (1, \infty)$, we assume for its conjugate $q = (1 - \frac{1}{p})^{-1} \in (1, \infty)$ that the space of all equivalent local martingale measures with L^q-integrable densities is nonempty, i.e. $\mathcal{M}_e^q \neq \emptyset$, where

$$\mathcal{M}_i^q = \{Q | dQ = Z_T dP,\ Z \in \mathcal{D}_i^q\} \subset L^q(P),\ q \in [1, \infty),\ i = e, a, S$$

with

$$\begin{aligned} \mathcal{D}_e^q &= \{Z \in \mathcal{U}^q | E(Z_T) = 1,\ Z_T > 0,\ SZ \in \mathrm{M}_{loc}\}, \\ \mathcal{D}_a^q &= \{Z \in \mathcal{U}^q | E(Z_T) = 1,\ Z_T \geq 0,\ SZ \in \mathrm{M}_{loc}\}, \\ \mathcal{D}_S^q &= \{Z \in \mathcal{U}^q | E(Z_T) = 1,\ SZ \in \mathrm{M}_{loc}\}, \end{aligned}$$

and $\mathcal{U}^q$ the class of uniformly integrable martingales M with $E^{\frac{1}{p}}(|M_T|^p) < \infty$. We set $\mathcal{M}_i^1 =: \mathcal{M}_i$. As we do not use the notion density process very frequently and if the notation is clear from the context, we write Z instead of Z_T and add a superscript to Z when denoting a density process. Further, define

$$\mathbb{P}_f(P) := \{Q \in \mathcal{M}_a : H(Q|P) < \infty)\}$$

with $H(Q|P) = E_P(\frac{dQ}{dP} \log(\frac{dQ}{dP}))$ if $Q \ll P$ and ∞ otherwise. If $\mathbb{P}_f(P)$ is non-empty, e.g. if condition (C) in [4, 6] holds (see section 4.1.1), then the minimal entropy martingale measure exists, i.e. the unique measure in $\mathbb{P}_f(P)$ that minimizes $H(Q|P)$. Under some minor assumptions it turns out that $\mathcal{M}_e^q \cap \mathbb{P}_f(P)$, $q = 1 + \tilde{\epsilon}$ for an $\tilde{\epsilon} > 0$, is the right choice for $\mathcal{A}^{\exp}$. To find a solution of the above dynamic problem (6), we have to find a suitable static problem. For $u_{2m}(x) = -(1 - \frac{\alpha x}{2m})^{2m}$, we solve the following one:

$$V(\tilde{x})_C \equiv \sup_{X \in \mathcal{O}^p, \forall Q \in \mathcal{M}_a^q E_Q(X) \leq \tilde{x}} E[U(X)],\ \tilde{x} \in \mathbb{R}, \tag{7}$$

where $\mathcal{O}^p := \{X \in L^p(\mathcal{F}_T, P) : EU(X) < \infty\}$, which is equal to L^p for the 2m-th problem $p = 2m$, $q = \frac{2m}{2m-1}$ its conjugate. Clearly, the supremum in (7) is not smaller than the one in (6), because for an $\vartheta \in \mathcal{A}^p : X = \int_0^T \vartheta dS \in L^p(\Omega, \mathcal{F}_T, P)$ and therefore $E_Q(X) < \infty$. It is well known that then, by an application of Burkholder-Davis-Gundy inequality, $\int_0^{\cdot} \vartheta dS$ is a Q-martingale, $E_Q(X) = 0$, see Grandits and Rheinländer [7]. In the exponential case ($u_{exp}(x) = -e^{-\alpha x}$), we show under a minor assumption that there exists an $\tilde{\epsilon} > 0$ such that the density of the minimal entropy martingale measure $Z_{\min}$ is in $L^{1+\tilde{\epsilon}}$. We can set $q = 1 + \tilde{\epsilon}$. A later construction of an optimal portfolio gives the necessary integrability of the optimal terminal value, i.e. $X_{expopt} \in \mathcal{O}^{p_0}$, $p_0 = (1 - \frac{1}{1+\tilde{\epsilon}})^{-1}$. In all cases, we set without loss of generality $\alpha = 1$.

Remark 2.1. *To obtain equality of the dynamic and static problem, $\mathcal{M}_a^q$ might not be large enough such that the constraints in (7) restrict the optimal claim X_0 so that it is attainable, i.e. $X_0 \in \{X : \exists Y \in \mathcal{W}_C \; Y_T = X_0\}$. Fortunately, if the density of the q-optimal measure is positive and has the form proposed in [10], we can prove equality, i.e. by finding a strategy that reaches X_0. See section 5.2. That means, we do not have to consider the set of signed measures $\mathcal{M}_S^q$ further. Signed measures are not considered in [10]. However, if it can be proven that the proposed form is the q-optimal signed measure, similar arguments can be used to find a hedging strategy. The static problem over the class of signed measures is the right choice: $\mathcal{M}_S^q$. As signed martingale measures struggle with a good interpretation in mathematical finance, we stick to cases where $\mathcal{M}_a^q$ is appropriate. More theoretical considerations about signed measures are left for future research.*

3. Solving the Static Problem

3.1. **Exponential Utility Function and its Approximating Sequence.** Next, we recall results from [13], in particular the solution to the static problem given in (7) with $u(X) = -e^{-\alpha x}$ and $u_{2m}(x) = -(1 - \frac{\alpha x}{2m})^{2m}$.

3.1.1. *Static solution of the 2m-th problem.* The procedure used in [13] does not use continuity of S, as long as sufficient integrability of the dual solution is ensured. Exactly as in the continuous case, we obtain for the 2mth problem (w.r.t the utility function $u_{2m}(x) = -(1 - \frac{x}{2m})^{2m}$) and for initial wealth $\tilde{x} \leq 2m$:

$$X_0^{(2m)}(\tilde{x}) = 2m - 2m \left(Z_{2m} \left(\frac{2m - \tilde{x}}{2mE\left(Z_{2m}^{\frac{2m}{2m-1}}\right)} \right)^{2m-1} \right)^{\frac{1}{2m-1}} \tag{8}$$

where $Z_{2m} = Z_T^{(2m)}$ and $Z^{(2m)}$ the solution of $\min_{Z \in \mathcal{D}_a^{\frac{2m}{2m-1}}} E(Z_T^{\frac{2m}{2m-1}})$. $X_0^{(2m)}$ is in L^{2m} ($q = \frac{2m}{2m-1}$, and therefore $p = \frac{q}{q-1} = 2m$), since

$$|X_0^{(2m)}(\tilde{x})|^{2m} = K_{\tilde{x}}(m) |Z_{2m}|^{\frac{2m}{2m-1}} < \infty \tag{9}$$

as $Z_{2m} \in L^q$, $q = \frac{2m}{2m-1}$ and $K_{\tilde{x}}(m)$ a constant depending on m and $\tilde{x}$.

3.1.2. *Static solution of the exponential problem.* If the minimal entropy measure $Q_{\min}$ with density $Z_{\min}$ is in $L^{1+\tilde{\epsilon}}$, we have:

$$X_0^{\exp}(\tilde{x}) \quad = \quad -\log Z_{\min} + H(Q_{\min}|P) + \tilde{x} \tag{10}$$

We postpone the proof that $X_0^{\exp}$ is in $\mathcal{O}^{p_0}$, $p_0 = \frac{1+\tilde{\epsilon}}{\tilde{\epsilon}}$, see Lemma 5.2. The aim of the following sections is to prove that the solutions of the above static problems are attainable. Furthermore, we show convergence of its dynamic equivalents, i.e. convergence of the optimal portfolios. This is done via the above duality, so it is necessary to study equivalent martingale measures:

4. Connections between q-optimal martingale measures and the minimal entropy martingale measure

Dual solutions to the above primal problem are q-optimal martingale measures for $q = \frac{2m}{2m-1}$ with density Z_{2m} and the minimal entropy measure with density $Z_{\min}$ as mentioned in section 3. Convergence of the dual solutions is therefore essential to show convergence of the $2m$ ths to the exponential problem. Jeanblanc et al. [10] treat q-optimal measures, give assumptions for its existence and prove its convergence in entropy. In the one dimensional case, we replace their assumptions on their convergence result by a minor monotonicity condition. This assumption is in fact satisfied in a lot of interesting cases concerning portfolio optimization. We show convergence in $L^{1+\tilde{\epsilon}}, \tilde{\epsilon} > 0$. In the multidimensional case, we show that Girsanov parameters $(\theta_q)_q$ are bounded, if the Brownian component is non zero, i.e. $\sigma\sigma' \geq \epsilon I_{n\times n}$. An integrability condition yields convergence in $L^{1+\tilde{\epsilon}}, \tilde{\epsilon} > 0$. The integrability condition is always satisfied if jumps are bounded from above. Note further, already a simple one dimensional jump diffusion model only makes sense, if $\sigma \neq 0$. Otherwise the model only consists of some rare jumps and arbitrage cannot always be excluded, (see Runggaldier [17]).

4.1. **Martingale Measures.**

We start with the minimal entropy martingale measure:

4.1.1. *The minimal entropy measure.* Recall, $Q_{\min}$ denotes the minimizer of $H(Q|P)$ in $\mathbb{P}_f(P)$. Frittelli [5] proves that if $\mathcal{M}_e \cap \mathbb{P}_f(P) \neq \emptyset$ and S is locally bounded (otherwise we need to optimize over the set of true martingale measures, i.e. S is a Q-martingale), then $Q_{\min}$ uniquely exists in $\mathcal{M}_e$. As we need the minimal entropy measure in explicit form, we propose a condition as in [4, 6], implying that S is already a $Q_{\min}$-martingale (local boundedness of S is not necessary!):

Assumption 4.1. *(C)*
There exists a vector $\theta_e \in \mathbb{R}^n$ that satisfies the following conditions:

(1)
$$\int \|(e^x - \mathbf{1})e^{\theta_e'(e^x-\mathbf{1})} - x1_{\|x\|\leq 1}\|\nu(dx) < \infty \tag{11}$$

(2)

$$\sigma\sigma'\theta_e + \int (e^x - \mathbf{1}) e^{\theta_e'(e^x - \mathbf{1})} - x 1_{\|x\|\leq 1} \nu(dx) = \beta \tag{12}$$

Note, in the one dimensional case, we just need $\int_{x>1} e^x e^{\theta_e(e^x-1)} \nu(dx) < \infty$. $x1_{\|x\|\leq 1}$ can be replaced by an arbitrary but fixed truncation function h. Setting $f_e = \theta_e'\sigma$ and $g_e(x) = \theta_e'(e^x - \mathbf{1})$, by [4, 6] the density of $Q_{\min}$ has the following form:

$$\begin{aligned} Z_t^{(\min)} &= e^{f_e W_t - \frac{1}{2} f_e f_e' t + \int_{(0,t]} \int_{\|x\|>1} g_e(x) N(dx,du) + \int_{(0,t]} \int_{\|x\|\leq 1} g_e(x) \tilde{N}(dx,du)} \\ &\quad e^{-t \int (e^{g_e(x)} - 1 - g_e(x) 1_{\|x\|\leq 1}) \nu(dx)} \end{aligned} \tag{13}$$

The following assumption provides a sufficient condition that $Z_{\min} \in L^{1+\tilde{\epsilon}}$

Assumption 4.2. *θ_e is an inner point of $\mathcal{K}$, where*

$$\mathcal{K} = \{\theta \in \mathbb{R}^n : \int \|(e^x - \mathbf{1}) e^{\theta'(e^x - \mathbf{1})} - h(x)\| \nu(dx) < \infty\}.$$

Clearly, if all jumps are bounded from above, we do not have to further care about assumption 4.2. At least in the one dimensional case assumption 4.2 is not very restrictive, because all examples provided by [6] satisfy this condition:

Example 4.1. *Setting $\beta_0 = \sup \mathcal{K}$ in the case of $n = 1$, the above condition means $\theta_e < \beta_0$. In [6] is shown that condition (C) is satisfied in several cases, e.g. for several kinds of variance gamma processes, stable processes, normal inverse gaussian Lévy processes (*Hubalek and Sgarra *[9]), and in particular for a compound Poisson process with support $(-\infty, L]$. Furthermore β_0 is either 0 or ∞. If $\theta_e = 0$, then $Q_{\min} = P$. If $\theta_e \neq 0$, $\beta_0 = 0$ or $\beta_0 = \infty$ so $\theta_e < \beta_0$. $\beta_0 = \infty$ holds for a Compound Poisson Process with support $(-\infty, L]$. For stable processes, depending on the parameter specification, β_0 can be shown to be 0 or ∞. For certain types of variance gamma and NIG processes β_0 is 0. Note, $\beta_0 = 0$ implies that if θ_e exists it has to be negative! In these cases, the sufficient condition for the existence of the q-optimal measure (C_q see below) given in [10] fails! When considering q-optimal measures, we therefore modify such models. In these models, we assume that jumps are bounded from above, so $\beta_0 = \infty$.*

As mentioned, we obtain:

Theorem 4.1. *Under assumption 4.1 and 4.2, the density of the minimal entropy martingale measure is in $L^{1+\tilde{\epsilon}}$ for an $\tilde{\epsilon} > 0$.*

Proof. Similar to the proof of Theorem B in [4]:

$$\int_{\|y\|>1} e^{\theta' y} \tilde{\nu}(dy) = \int_{\|x\|>K} e^{\theta'(e^x - \mathbf{1})} \nu(dx) < \infty$$

for all $\theta \in \mathcal{K}$ and a constant $K > 0$. $\tilde{\nu}$ denotes the Lévy measure of $\tilde{L}$. The assertion follows by Theorem 25.17. in Sato [18] and assumption 4.2. □

We continue with the q-optimal measure:

4.1.2. *q-optimal martingale measure.* As in [10] we assume,

Assumption 4.3. *(C_q)*
There exists an $\theta_q \in \mathbb{R}^n$ with

$$\mathrm{eg}_q(x) := ((q-1)\theta_q'(e^x - \mathbf{1}) + 1)^{\frac{1}{q-1}} > 0 \ \ \nu\text{-a.s.}$$

well defined such that $\int_{\mathbb{R}^n}(\mathrm{eg}_q^q(x)) - 1 - q(\mathrm{eg}_q(x) - 1)\nu(dx) < \infty$ *and*

$$\sigma\sigma'\theta_q + \int (e^x - \mathbf{1})\mathrm{eg}_q(x) - x1_{\|x\|\leq 1}\nu(dx) = \beta \tag{14}$$

Jeanblanc et al. [10] show that (C_q) is a sufficient condition for the existence of the equivalent q-optimal measure and provide an explicit form of its density: By defining

$$g_q(x) = \log \mathrm{eg}_q(x), \ f_q = \theta_q'\sigma, \tag{15}$$

they obtain

$$\begin{aligned} Z_t(f_q, g_q) &= e^{f_q W_t - \frac{t}{2} f_q f_q' + \int_0^t \int_{\|x\|>1} g_q(x) N(dx,ds) + \int_0^t \int_{\|x\|\leq 1} g_q(x) \tilde{N}(dx,ds)} \\ &\quad \times e^{-t\int (e^{g_q(x)} - 1 - g_q(x) 1_{\|x\|\leq 1})\nu(dx)}. \end{aligned} \tag{16}$$

(14) implies that $Z(f_q, g_q)$ is the density process of a positive local martingale measure, see e.g. Kunita [15]. By Theorem 2.9. in [10], it is the q-optimal martingale measure, denoted by $Q^{(q)}$.

4.1.3. *Convergence to the minimal entropy measure.* Next, we have to clarify when the q-optimal measures actually converge to the minimal entropy measure when q tends to 1. In general, we assume (C_q) for all $q \in (1, q_0]$ for an $q_0 > 1$ and that $\sigma\sigma' \geq \epsilon I_{n\times n}$ for an $\hat{\epsilon} > 0$. We next show that $(\theta_q)_{q\in[1,q_0]}$ is bounded, an essential assumption proposed in [10]. In the multidimensional case, we therefore need to propose the following integrability condition:

Assumption 4.4. *Define with* $h(x) = x1_{\|x\|\leq 1}$:

$$\mathcal{K}^n = \{K \in \mathbb{R} : \int \|(e^x - \mathbf{1})e^{K\sum|(e^{x_i} - \mathbf{1})|} - h(x)\|\nu(dx) < \infty\} \tag{17}$$

Assume

1. *There exists an $\hat{\epsilon} > 0$ such that $\sigma\sigma' \geq I_{n\times n}\hat{\epsilon}$*
2. $R := \hat{\epsilon}^{-1}\|\int(e^x - \mathbf{1}) - x1_{\|x\|\leq 1}\nu(dx) - \beta\| \in \mathcal{K}^n$

Lemma 4.2. *We have*

1. *Suppose all integrals in (14) are well defined and there exists an n-dimensional vector θ_q solving (14) resp. condition (C) holds. Under assumption 4.4.1:* $\|\theta_q\| \leq R$ resp. $\|\theta_e\| \leq R$
2. *Suppose there exists a $q_0 > 1$ such that C_q is satisfied for all $q \in [1, q_0]$, condition (C) and assumption 4.4 hold then $\theta_q \to \theta_e$.*

Proof. If $\int(e^x - \mathbf{1}) - x1_{\|x\|\le 1}\nu(dx) - \beta = \mathbf{0}$, $\theta_e = \theta_q$ all measures coincide. Excluding this case, we define as in [10]

$$\begin{aligned}\Phi(q,\theta) &= -\beta + \sigma\sigma'\theta + \int(e^x - \mathbf{1})((q-1)\theta'(e^x - \mathbf{1}) + 1)^{\frac{1}{q-1}} - x1_{\|x\|\le 1}\nu(dx) \\ \Phi_e(\theta) &= \Phi(e,\theta) = -\beta + \sigma\sigma'\theta + \int(e^x - \mathbf{1})e^{\theta'(e^x-\mathbf{1})} - x1_{\|x\|\le 1}\nu(dx)\end{aligned}$$

The equation defining θ_a is $\Phi(a,\theta_a) = 0$, for $a = q, e$.

1: We show boundedness of θ_a. For an arbitrary θ in the domain of Φ, we have

$$\begin{aligned}\theta'\Phi(q,\theta) &= -\theta'\beta + \theta'\sigma\sigma'\theta \\ &\quad + \int\theta'(e^x - \mathbf{1})((q-1)\theta'(e^x - \mathbf{1}) + 1)^{\frac{1}{q-1}} - \theta' x1_{\|x\|\le 1}\nu(dx) \\ &\ge \theta'(\int(e^x - \mathbf{1}) - x1_{\|x\|\le 1}\nu(dx) - \beta) + \theta'\sigma\sigma'\theta \\ &\ge -\|\theta\|\|\int(e^x - \mathbf{1}) - x1_{\|x\|\le 1}\nu(dx) - \beta\| + \hat{\epsilon}\|\theta\|^2\end{aligned}$$

by Cauchy-Schwartz-inequality, analogous for $e^{\theta'(e^x-\mathbf{1})}$ instead of $((q-1)\theta'(e^x - \mathbf{1}) + 1)^{\frac{1}{q-1}}$. As $\Phi(a,\theta_a) = \mathbf{0}$ implies $\theta_a'\Phi(a,\theta_a) = 0$,

$$\begin{aligned}0 &\ge -\|\theta_a\|\|\int(e^x - \mathbf{1}) - x1_{\|x\|\le 1}\nu(dx) - \beta\| + \hat{\epsilon}\|\theta_a\|^2 \\ \|\theta_a\| &\le \hat{\epsilon}^{-1}\|\int(e^x - \mathbf{1}) - x1_{\|x\|\le 1}\nu(dx) - \beta\| =: R\end{aligned}$$

θ_e, θ_q are elements of the compact ball in $\mathbb{R}^n$ with radius R around zero.

2: As condition (C_q) holds, $((q-1)\theta'(e^x - \mathbf{1}) + 1)^{\frac{1}{q-1}}$ is monotonically decreasing in q and for $x_i > 0$,

$$\begin{aligned}&(e^{x_i} - 1)((q-1)\theta'(e^x - \mathbf{1}) + 1)^{\frac{1}{q-1}} - x_i 1_{\|x\|<1} \\ \le\ &(e^{x_i} - 1)e^{\theta'(e^x-\mathbf{1})} - x_i 1_{\|x\|<1} \\ \le\ &(e^{x_i} - 1)e^{R\sum_i |e^{x_i}-1|} - x_i 1_{\|x\|<1} = (**) \end{aligned} \tag{18}$$

which is finite because R is in $\mathcal{K}^n$. On the other hand for $x_i > 0$

$$\begin{aligned}&(e^{x_i} - 1)((q-1)\theta'(e^x - \mathbf{1}) + 1)^{\frac{1}{q-1}} - x_i 1_{\|x\|\le 1} \\ \ge\ &(e^{x_i} - 1)(\theta'(e^x - \mathbf{1}) + 1) - x_i 1_{\|x\|\le 1} \\ =\ &e^{x_i} - 1 - x_i 1_{\|x\|\le 1} + (e^{x_i} - 1)\theta'(e^x - \mathbf{1}) \\ \ge\ &e^{x_i} - 1 - x_i 1_{\|x\|\le 1} - (e^{x_i} - 1)\|\theta\|\|e^x - \mathbf{1}\| \\ \ge\ &e^{x_i} - 1 - x_i 1_{\|x\|\le 1} - R\|e^x - \mathbf{1}\|^2 = (*)\end{aligned} \tag{19}$$

If $x_i < 0$, replace $-R\|e^x - \mathbf{1}\|^2$ by $R\|e^x - \mathbf{1}\|^2$ then all inequalities in (18) and (19) turn around. So

$$|(e^{x_i} - 1)((q-1)\theta'(e^x - \mathbf{1}) + 1)^{\frac{1}{q-1}} - x_i 1_{\|x\|<1}| \quad \leq \quad |(*)| + |(**)|.$$

The last term is integrable independently of q and θ. So finally, we can change limit and integration in $\Phi(q,\theta)$. From basic analysis, we know that $\lim_{n\to\infty}(1 + x_n/n)^n = e^x$, for every sequence x_n with $x_n \to x$. Pick an arbitrary cluster point of $(\theta_q)_q$. It has to be finite, because $(\theta_q)_q$ are bounded. Then for the corresponding subsequence $(\theta_{q_r})_r$, q_r tending to 1, $\lim_r \theta_{q_r}$ exists. As we can change limit and integration $0 = \lim_{q_r\to 1}\Phi(q_r, \theta_{q_r}) = \Phi(e, \lim_r \theta_{q_r})$. Since the minimal entropy measure uniquely exists under assumption 4.1 $\lim_r \theta_{q_r} = \theta_e$ for every subsequence with $q_r \to 1$. Hence, $\lim \theta_q = \theta_e$. □

Remark 4.1. *In the one dimensional case, we are able to remove the stronger integrability condition by a monotonicity argument. We propose*

Assumption 4.5. (C_q) *for all* $q \in (1, q_0]$ *for an* $q_0 > 1$ *and condition (C). There exists an* $q_1 > 1$ *such that for all* $1 < q \leq q_1$ *and for all* $\theta \geq 0$ *(if* $K_0 < 0$*) resp.* $\theta \leq 0$ *(if* $K_0 = \Phi(q,0) = \Phi(e,0) > 0$*) in the domain of* $\Phi(q,\cdot)$, $\Phi(q,\theta)$ *is monoton in the same direction, i.e. either decreasing or increasing for all* $q \leq q_1$.

The last assumption is e.g. satisfied, if only positive or only negative jumps appear. Secondly, if $\tilde{\nu}(A) = \int 1_{e^x - 1 \in A}\nu(dx)$ *is symmetric around zero and* $e^{x_i} - 1 \in (-1, L)$, $L = 1$. *Further shifts to the negative side are allowed. A Brownian component is not necessarily needed.* $\theta_q \to \theta_e$ *follows from the fact that* $\Phi(q,\theta)$ *is increasing in* θ *around* θ_q.

Finally,

Lemma 4.3. *Assume (C),* C_q *for all* $q \in (1, q_0]$ *for an* $q_0 > 1$, *condition 2.2, and 4.2/4.5 or 4.4, respectively. Then there exists an* $\tilde{\epsilon} > 0$ *such that*

$$\sup_{1<q\leq q_0} E((Z_{(q)}^{1+\tilde{\epsilon}})) < \infty.$$

In particular, $(Z_{(q)}^{1+\tilde{\epsilon}})_{q\in(1,q_0]}$ *is uniformly integrable.*

Lemma 4.4. *Under the assumptions of Lemma 4.3, we obtain*

$$\log Z_q \xrightarrow{L^2} \log Z_{\min},\ Z_q \xrightarrow{L^{1+\tilde{\epsilon}}} Z_{\min}$$

In particular, $E(\log Z_q) \to E(\log Z_{\min})$.

Recall, both sets of assumptions imply that $\theta_q \to \theta_e$. We start with the proof of Lemma 4.3:

Proof. (Proof of Lemma 4.3) Set $1+\tilde{\epsilon} =: \kappa$.

$$\begin{aligned} Z^{\kappa}_{t,(q)} &= \exp(\kappa f_q W_t - \kappa \frac{1}{2} f_q f_q' t - t\kappa \int (e^{g_q(x)} - 1 - g_q(x) 1_{\|x\|\leq 1})\nu(dx)) \\ &\quad \times \exp(\int_{(0,t]}\int_{\|x\|>1} \kappa g_q(x) N(dx,du) + \int_{(0,t]}\int_{\|x\|\leq 1} \kappa g_q(x)\tilde{N}(dx,du)) \end{aligned}$$

$$= \exp(\kappa f_q W_t - \kappa^2 \frac{1}{2} f_q f_q' t) \tag{20}$$

$$\times \exp(\int_{(0,t]}\int_{\|x\|>1} \kappa g_q(x) N(dx,du) + \int_{(0,t]}\int_{\|x\|\leq 1} \kappa g_q(x)\tilde{N}(dx,du))$$

$$\times \exp(-t\int (e^{\kappa g_q(x)} - 1 - \kappa g_q(x) 1_{\|x\|\leq 1})\nu(dx)) \tag{21}$$

$$\times \exp(t\int (e^{\kappa g_q(x)} - 1) - \kappa(e^{g_q(x)} - 1)\nu(dx)) \tag{22}$$

$$\times \exp(-\kappa \frac{1}{2} f_q f_q' t + \kappa^2 \frac{1}{2} f_q f_q' t) \tag{23}$$

We can treat the Brownian part and the jump part independently. (20) is a martingale, because all coefficients are constants (Novikov's condition). (23) is deterministic and therefore finite for $t \leq T$. It is independent of q because $\theta_q \in (\theta_e - \epsilon, \theta_e + \epsilon)/\|\theta_q\| \leq R$. The stochastic exponential of a martingale, which is a Lévy process is again a martingale. So it remains to show that $M_q = \int (e^{\kappa g_q(x)} - 1)\tilde{N}(dx,ds)$ is a martingale. It is a Lévy process with characteristic triplet $(0, \gamma, \nu_f)$, where $\nu_f(G) = \int 1_{e^{\kappa g_q(x)}-1 \in G}\nu(dx)$ and $\gamma = \int_{\|x\|>1} x\nu_f(dx)$ (note $\int (e^{\kappa g_q(x)} - 1)^2 \nu(dx) < \infty$, because $|e^{\kappa g_q(x)} - 1| \leq \kappa|\theta_q'(e^x - \mathbf{1})|$ and $e^{\kappa g_q(x)} - 1$ is of the same order as $\kappa g_q(x)$ for $|\kappa g_q(x)| < 1$). If $\int_{|\kappa g_q(x)|>1} (e^{\kappa g_q(x)} - 1)\nu(dx)$ is finite, M_q is a martingale and so also (21). Since $(1 + \frac{y}{m})^m$ is increasing in m :

$$\begin{aligned} & \int (e^{g_q(x)} - 1 - \theta_q' x 1_{\|x\|<1})\nu(dx) \\ = & \int (1 + (q-1)\theta_q'(e^x - \mathbf{1}))^{1/(q-1)} - 1 - \theta_q' x 1_{\|x\|<1}\nu(dx) \\ \geq & \int (1 + \theta_q'(e^x - \mathbf{1})) - 1 - \theta_q' x 1_{\|x\|<1}\nu(dx) \\ = & \ \theta_q' \int e^x - \mathbf{1} - x 1_{\|x\|<1}\nu(dx) \end{aligned}$$

Set $K_0^{(i)} = \int e^{x_i} - 1 - x_i 1^{(i)}_{\|x\|<1}\nu(dx)$. Then componentwise

$$\theta_q^{(i)} \int e^{x_i} - 1 - x_i 1^{(i)}_{\|x\|<1}\nu(dx) \quad \geq \quad \theta_{e,\epsilon}^{(i)} \int e^{x_i} - 1 - x_i 1^{(i)}_{\|x\|<1}\nu(dx),$$

where $\theta^{(i)}_{e,\epsilon} = \theta^{(i)}_e \mp \epsilon$ if $K^{(i)}_0 >_< 0$. Further, for q close enough to 1, we have

$$\begin{aligned} &\int (e^{\kappa g_q(x)} - 1 - \kappa\theta'_q x 1_{\|x\|<1})\nu(dx) \\ = &\int (1 + (q-1)\frac{1}{\kappa}\kappa\theta'_q(e^x - \mathbf{1}))^{\kappa/(q-1)} - 1 - \kappa\theta'_q x 1_{\|x\|<1}\nu(dx) \\ \leq &\int \exp(\kappa\theta'_q(e^x - \mathbf{1})) - 1 - \kappa\theta'_q x 1_{\|x\|<1}\nu(dx). \end{aligned} \tag{24, 25}$$

We know that there exists a $\zeta \in (0,1)$ such that $\exp(x) = 1 + x + \frac{\exp(\xi)}{2!}x^2$, $\xi = \zeta x$. Hence, for $\|x\| \leq 1$ and $\zeta \in (0,1)$:

$$\begin{aligned} & \exp(\kappa\theta'_q(e^x - \mathbf{1})) - 1 - \kappa\theta'_q x \\ = \; & \exp(\zeta\kappa\theta'_q(e^x - \mathbf{1}))(\kappa\theta'_q(e^x - \mathbf{1}))^2/2 + \kappa\theta'_q(e^x - \mathbf{1} - x) \\ \leq \; & \max\{1, e^{\kappa\theta'_q(e^x - \mathbf{1})}\}(\kappa\theta'_q(e^x - \mathbf{1}))^2/2 + \kappa\theta'_q(e^x - \mathbf{1} - x). \end{aligned} \tag{26}$$

The last term can be estimated independently of q, because $\|x\| \leq 1$ and by assumption 4.2/4.4. Furthermore, components of θ_q are bounded by the corresponding components of $\theta_e \pm \epsilon$ or R, respectively. So the remaining assertions follow by assumption 2.2 and 4.2/4.4, see also the proof of Lemma 4.2. Finally, with a constant $K > 0$ independent of q:

$$E(Z^\kappa_{T,(q)}) = K_q e^{T\int (e^{\kappa g_q(x)} - 1) - \kappa(e^{g_q(x)} - 1)\nu(dx)} \leq K < \infty \tag{27}$$

Uniform integrability follows by the de la Vallé-Poisson Theorem as in [13]. □

Proof. (Proof of Lemma 4.4): By (13)/(16) and (15), we have

$$\begin{aligned} \log Z_q &= f_q W_T - \frac{1}{2} f_q f'_q T - T\int (e^{g_q(x)} - 1 - g_q(x))\nu(dx) \\ &\quad + \int_0^T \int g_q(x)\tilde{N}(dx, ds) \\ \log Z_{\min} &= f_e W_T - \frac{1}{2} f_e f'_e T - T\int (e^{g_e(x)} - 1 - g_e(x))\nu(dx) \\ &\quad + \int_0^T \int g_e(x)\tilde{N}(dx, ds). \end{aligned}$$

Since $|g_q(x)| \leq |(\theta_e \pm \epsilon\mathbf{1})'(e^x - \mathbf{1})|$, we have $\int |g_q(x)|^2\nu(dx) < \infty$ and $\int |g_e(x)|^2\nu(dx) < \infty$ by assumption 4.2. Hence, $\int_0^t \int g_q(x)\tilde{N}(dx, ds)$ and $\int_0^t \int g_e(x)\tilde{N}(dx, ds)$ are martingales. For $\|(\theta_e \pm \epsilon\mathbf{1})'(e^x - \mathbf{1})\| < 1$, $(e^{g_q(x)} - 1 - g_q(x))$ is of order $|g_q(x)|^2$ therefore dominated by a function independent of q (assumption 2.2), see (26). For $\|(\theta_e \pm \epsilon\mathbf{1})'(e^x - \mathbf{1})\| \geq 1$, by (25) and because $(\theta^{(i)}_q \in (\theta^{(i)}_e - \epsilon, \theta^{(i)}_e + \epsilon))$, $e^{g_q(x)} - 1$ is dominated by a function independent of q and $|g_q(x)|$ is dominated by $|(\theta_e \pm \epsilon\mathbf{1})'(e^x - \mathbf{1})|$, which is integrable by assumption 2.2 since x is bounded away from 0. By isometry and because g is deterministic, we have

$$E|\int_0^T \int (g_q(x) - g_e(x))\tilde{N}(dx, ds)|^2 = T \int (g_q(x) - g_e(x))^2 \nu(dx) \to 0$$

as $|g_q(x)|^2 \leq \|\theta_e \pm \mathbf{1}\epsilon\| \|e^x - \mathbf{1}\|$, $f_q \to f_e$, and $e^{g_q(x)} \to e^{g_e(x)}$. The last assertion holds because $f_q = \theta_q' \sigma$, $e^{g_q(x)} = (1+(q-1)\theta_q'(e^x - \mathbf{1}))^{1/(q-1)}$, $\theta_q \to \theta_e$, and $\lim_{n\to\infty}(1 + x_n/n)^n = e^x$, for every sequence x_n with $x_n \to x$. We have convergence in L^2 for $\log Z_q$ and therefore convergence in probability of Z_q to $Z_{\min}$. From Lemma 4.3 the convergence of $Z_q = Z_T^{(q)}$ also holds in $L^{1+\tilde{\epsilon}}$ for an $\tilde{\epsilon} > 0$ and so $Z_{\min} \in L^{1+\tilde{\epsilon}}$. This follows directly from uniform integrability and the convergence in probability. □

5. Convergence of the Exponential Utility Problem

In the continuous case, Kohlmann and Niethammer [13] at first show L^1-convergence of the terminal values $X_0^{(2m)}$ to $X_0^{(\exp)}$ in a quite general setting using the convergence of the q-optimal measures to the minimal entropy martingale measure. If in addition the deterministic trade off process is deterministic, MEMM coincides with all qMMM. In this case, they can show convergence of the portfolio in L^r-supremums norm ($r \leq 1$). Furthermore, under this additional stronger assumption, it would not have been necessary to show convergence of the terminal values, separately. In our case, as long as assumption 2.2 is satisfied, the trade off process is deterministic. However, for Lévy processes q-optimal measures do not coincide with the minimal entropy measure. Even though we can prove convergence of the portfolios which again implies L^r-convergence of the optimal terminal values. The proof of the convergence of the portfolios distinguishes a lot from the continuous case, whereas the proof regarding the terminal values follows almost the same steps as in [13] with some different technical details. The reader may skip section 5.1 - the convergence of the terminal values and continue with the portfolios. However, to give some room for future research, we draw the connection to [13]:

5.1. Convergence of the Terminal Values and the Value Functions.

The Convergence of the terminal values of the 2mth problem to the exponential one in [13] heavily relies on the continuity assumption of S. Fortunately, we are able to overcome this problem. In this section, we go through the proof given in [13], explain where problems appear in the discontinuous case and solve them in a Lévy setting.

Assume (C_q) for all $q \in (1, q_0)$ for an $q_0 > 1$ and condition (C) implying that (equivalent) qMMMs and MEMM exist. To establish L^1-convergence of $X_0^{(2m)}$ to $X_0^{\exp}$, we need to prove the same three steps as in [13]. For every positive, real sequence $(y_m)_m$ with limit y:

(1) a) $(Z_{2m}2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}))_m$ is uniformly integrable.
b) $2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}})_m$ is uniformly integrable.
(2) $Z_{2m} \xrightarrow{L^1} Z_{\min}$, $m \to \infty$, $(Z_{2m} := Z_T^{(\frac{2m}{2m-1})})$
(3) $y_m Z_{2m} 2m(1-(Z_{2m}y_m)^{\frac{1}{2m-1}}) \xrightarrow{L^1} -yZ_{\min}\log(Z_{\min}y)$

The proof of Item 3 in [13] does not rely on the continuity assumption, we therefore do not have to prove this item again. For item 1b, it remains to show that $E(\log Z_{2m}) \to E(\log Z_{\min})$, see Lemma 4.4. To establish item 1a, we only need uniform integrability of $(Z_{(q)}^{1+\tilde{\epsilon}})_q$, $\tilde{\epsilon} > 0$, see Lemma 4.3. Item 2 follows from 1a and 1b, see again Lemma 4.4. It should be noted, that item 1b uses convergence of the Girsanov parameters which was firstly shown in [10], but under different assumptions. Finally,

Theorem 5.1. *In our model with an exponential Lévy process $S = S_0 + M + A$ given, let one of the following sets of assumptions be satisfied:*

(1) $n = 1$, *assumptions 2.2, 4.2, and 4.5*
(2) *Assumptions (C), C_q for all $q \in (1, q_0]$ for an $q_0 > 1$, 2.2, 4.4*
(3) *X is a piecewise constant real valued Lévy process with jumps lower than a constant $L < \infty$ plus a Brownian motion with $\sigma \neq 0$.*

Then, the solution of $2m$-th problem converges in L^1 to the solution of the exponential problem, i.e. $X_0^{(2m)}(\tilde{x}) \xrightarrow{L^1} X_0^{exp}(\tilde{x})$.
The values of the dual and primal problems converge.

■

Under the third assumption the second is satisfied, see also the example in [10].

5.2. **Convergence to the Optimal Portfolio.** Next we clarify when the optimal portfolios $\vartheta^{(2m)}$ of the $2m$-problems converge to the optimal portfolio $\vartheta^{(exp)}$ of the exponential problem. We solve the pricing equation of the claim $X_0^{(2m)}(\tilde{x})$:

$$\begin{aligned} dp_t^{(2m),\tilde{x}} &= (\vartheta_t^{(2m)})' d\langle M\rangle_t \hat{\lambda}_t + (\vartheta_t^{(2m)})' dM_t + d\check{M}^{(2m)}, \qquad (28) \\ p^{(2m),\tilde{x}}(T) &= X_0^{(2m)}(\tilde{x}) \end{aligned}$$

where $\check{M}^{(2m)}$ is the orthogonal term appearing in the Föllmer-Schweizer-decomposition. We are able to find a portfolio such that $\check{M}^{(2m)}$ is zero, i.e. $X_0^{(2m)}$, the optimal solution of the static problem, is attainable. Next, we give a short motivation for the solution $(p^{(2m),\tilde{x}}, \vartheta^{(2m)})$ of (28). A reader not interested in this motivation can skip the next paragraph. $(p^{(2m),\tilde{x}}, \vartheta^{(2m)})$ is given in Lemma 5.2:

5.2.1. *Motivation.* As in the continuous case a possible candidate for the solution is

$$\begin{aligned}\tilde{p}_t^{(2m),\tilde{x}} &:= \hat{Z}_t^{-1}E(\hat{Z}_T X_0^{(2m)}(\tilde{x})|\mathcal{F}_t) \\ &= \hat{Z}_t^{-1}E(\hat{Z}_T 2m(1-Z_{2m}^{\frac{1}{2m-1}}(1-\frac{\tilde{x}}{2m})(E(Z_{2m}^{\frac{2m}{2m-1}}))^{-1})|\mathcal{F}_t),\end{aligned} \tag{29}$$

where with $\mathrm{eg}_*(x) = \mu_*'(e^x-1)+1$, $g_*(x) = \log(\mathrm{eg}_*(x))$ (if well defined) and $f_2 = \mu_*'\sigma$

$$\begin{aligned}\hat{Z}_t &= e^{\mu_*\sigma W_t - \frac{1}{2}\mu_*'\sigma\sigma'\mu_*'t + \int_0^t\int_{\|x\|>1} g_*(x)N(dx,ds) + \int_0^t\int_{\|x\|\le 1} g_*(x)\tilde{N}(dx,ds)} \\ &\quad \times e^{-t\mu_*'\int(e^x-1-g_*(x)1_{\|x\|\le 1})\nu(dx)}.\end{aligned}$$

However, $\hat{Z}$ is the density of the minimal and the variance optimal martingale measure (under our conditions both are equal and $g_2(x) = g_*(x)$), only if its density is positive. Otherwise $g_*(x)$ is not defined. That would restrict the set of processes enormously, see example 5.1 in the end. Actually, we do not have to assume that the variance optimal measure is not signed. Fortunately when heuristically calculating $\tilde{p}_t^{(2m),\tilde{x}}$, all terms containing $g_*(x)$ cancel out. The version obtained from this calculation, denoted by $p_t^{(2m),\tilde{x}}$ see Lemma 5.2, satisfies $p_T^{(2m),\tilde{x}} = X_0^{(2m)}(\tilde{x})$. Under assumption 2.2 all integrals exist. Itô's formula and a coefficient comparison then yield that this transformed process $p^{(2m),\tilde{x}}$ is in fact equal to the optimal price process $Y^{(2m),\tilde{x}}$, that reaches $X_0^{(2m),\tilde{x}}$. If the variance optimal measure exists and is equivalent to P: $\tilde{p}_t^{(2m),\tilde{x}} = p_t^{(2m),\tilde{x}}$.

We start to derive $E(Z_q^q)$, (e.g. see proof of Lemma 4.3 setting $\kappa = q$) as before:

$$E(Z_q^q) = \exp(\frac{1}{2}q(q-1)f_q f_q' T + \int e^{qg_q(x)} - 1 - q(e^{g_q(x)}-1)\nu(dx)T),$$

where $Z_q^q = Z_{2m}^{\frac{2m}{2m-1}}$, with $q = \frac{2m}{2m-1}$. Next, we need to deal with $E(\hat{Z}_T Z_q^{q-1})$:

$$E(\hat{Z}_T Z_q^{q-1}|\mathcal{F}_t) = E(\tilde{M}_T^{(q)}|\mathcal{F}_t)\tilde{A}_T^q \tag{30}$$

where

$$\begin{aligned}\tilde{M}_t^{(q)} &= e^{\int_0^t\int_{\|x\|>1} g_*(x)+(q-1)g_q(x)N(dx,ds) + \int_0^t\int_{\|x\|\le 1} g_*(x)+(q-1)g_q(x)\tilde{N}(dx,ds)} \\ &\quad \times e^{(f_2+f_q(q-1))W_t - \frac{1}{2}t(f_2+(q-1)f_q)(f_2+(q-1)f_q)'} \\ &\quad \times e^{-t\int(\mathrm{eg}_*(x)e^{(q-1)g_q(x)} - 1 - (g_*(x)+(q-1)g_q(x))1_{\|x\|\le 1})\nu(dx)}\end{aligned}$$

$$\begin{aligned}\tilde{A}_T^{(q)} &= e^{-T\int((\mathrm{eg}_*(x)-1)+(q-1)(e^{g_q(x)}-1)-(\mathrm{eg}_*(x)e^{(q-1)g_q(x)}-1))\nu(dx)} \\ &\quad \times \exp(f_2 f_q(q-1)T + \frac{1}{2}((q-1)f_q)((q-1)f_q)'T - \frac{1}{2}(q-1)f_q f_q T)\end{aligned}$$

$\tilde{M}^{(q)}$ is a martingale, because $\int_{\|x\|>1}|(\mathrm{eg}_*(x)e^{(q-1)g_q(x)}-1)|\nu(dx) < \infty$ by (15) and (33) below and similar arguments as in the proof of Lemma 4.3.

With (30), we have

$$\frac{E(\hat{Z}_T Z_q^{q-1}|\mathcal{F}_t)}{E(Z_q^q)} = \tilde{M}_t^{(q)} \tilde{A}_T^{(q)} (E(Z_q^q))^{-1} = \tilde{M}_t^{(q)} A_T^{(q)}, \qquad (31)$$

where

$$\begin{aligned} A_T^{(q)} &= e^{-T\int((\mathrm{e}g_*(x)-1)+(q-1)(e^{g_q(x)}-1)-(\mathrm{e}g_*(x)e^{(q-1)g_q(x)}-1))\nu(dx)} \\ &\quad \times e^{f_2 f_q'(q-1)T-(q-1)f_q f_q' T} e^{-\int e^{qg_q(x)}-1-q(e^{g_q(x)}-1)\nu(dx)T} \\ &= e^{-T\int((\mathrm{e}g_*(x)-1)-(e^{g_q(x)}-1)-(\mathrm{e}g_*(x)e^{(q-1)g_q(x)}-1)+(e^{qg_q(x)}-1))\nu(dx)} \\ &\quad \times \exp(f_2 f_q'(q-1)T - \frac{1}{2} q(q-1) f_q f_q T). \end{aligned}$$

So,

$$\begin{aligned} &\hat{Z}_t^{-1} \frac{E(\hat{Z}_T Z_q^{q-1}|\mathcal{F}_t)}{E(Z_q^q)} \\ &= \tilde{M}_t^{(q)} A_T^{(q)} e^{-\int_0^t g_*(x)N(dx,ds)+t\int(\mathrm{e}g_*(x)-1)\nu(dx)-f_2 W_t+\frac{1}{2}f_2 f_2' t} \\ &= M_t^{(q)} A_T^{(q)} \beta_t^{(q)}, \end{aligned}$$

where

$$\beta_t^{(q)} = e^{t\int(\mathrm{e}g_*(x)-1)+(e^{(q-1)g_q(x)}-1)-(\mathrm{e}g_*(x)e^{(q-1)g_q(x)}-1)\nu(dx)} e^{-f_2 f_q'(q-1)t}$$

and

$$\begin{aligned} M_t^{(q)} &= e^{\int_0^t \int_{\|x\|<1}(q-1)g_q(x)N(dx,ds)+\int_0^t\int_{\|x\|\le 1}(q-1)g_q(x)\tilde{N}(dx,ds)} \\ &\quad \times e^{f_q(q-1)W_t-\frac{1}{2}f_q f_q'(q-1)^2 t - t\int(e^{(q-1)g_q(x)}-1-(q-1)g_q(x)1_{\|x\|\le 1})\nu(dx)}, \end{aligned}$$

which is again a martingale by (15) and the arguments we had before. Furthermore, $A_T^{(q)} = 1$ using (31) because $E_Q(X_0^{(2m)})$ coincides for all $Q \in \mathcal{M}_a^q$, if $X_0^{(2m)}$ can be hedged which is shown next:

5.2.2. *Optimal portfolio of the 2mth problem and its convergence.* The next lemma provides the solution $(p^{(2m),\tilde{x}}, \vartheta^{(2m)})$ of BSDE (28):

Lemma 5.2. *Under condition C_{q_m}, $q_m = \frac{2m}{2m-1}$ for all $m \geq m_1$ for an $m_1 > 0$ and assumptions 2.2 and 4.2/4.5 or 4.4, there exists an $m_0 \geq m_1$ such that for all $m \geq m_0$ and $\tilde{x} \leq 2m$ the optimal wealth process of the 2m th problem has the following form:*

$$\begin{aligned} p_t^{(2m),\tilde{x}} - p_0^{(2m),\tilde{x}} &= 2m - (2m-\tilde{x}) M_t^{(q_m)} \beta_t^{(q_m)} \\ &= \tilde{x} + \int_0^t \vartheta_u^{(2m)} d\langle M\rangle_u \hat{\lambda}_u + \int_0^t \vartheta_u^{(2m)} dM_u \\ &= \tilde{x} + \int_0^t \vartheta_u^{(2m)} dS_u \end{aligned}$$

where $\vartheta_u^{(2m)} = -\frac{2m-\tilde{x}}{2m-1} M_{u-}^{(q_m)} \beta_u^{(q_m)} \theta_{q_m}' \mathbf{S}_{u-}^{-1}$ the portfolio process with $\vartheta^{(2m)} \in \mathcal{A}^{2m}$. Finally, $p_T^{(2m),\tilde{x}} = X_0^{(2m)}(\tilde{x})$.

Proof. The condition $p_T^{(2m),\tilde{x}} = X_0^{(2m)}(\tilde{x})$ is easily checked. For convenience we write $q_m = q$ and p its conjugate. We describe $p_t^{(2m),\tilde{x}}$ as a function $\tilde{f}_{2m}$ and apply Itô's formula:

$$p_t^{(2m),\tilde{x}} = -(2m-\tilde{x})A_T^{(q)} e^{V_t^{(2m)}} =: \tilde{f}_{2m}(V_t^{(2m)}) \tag{32}$$

where

$$\begin{aligned} V_t^{(2m)} &= \int_0^t \int_{\|x\|>1} (q-1)g_q(x)N(dx,ds) + \int_0^t \int_{\|x\|\le 1} (q-1)g_q(x)\tilde{N}(dx,ds) \\ &\quad + \int (\mathrm{eg}_*(x) - (\mathrm{eg}_*(x)e^{(q-1)g_q(x)} - (q-1)g_q(x)1_{\|x\|\le 1})\nu(dx) \\ &\quad - \frac{1}{2} f_q f_q'(q-1)^2 t - t f_2 f_q'(q-1) + f_q(q-1)W_t. \end{aligned}$$

Define $G_t^{(q)} = (2m-\tilde{x})M_t^{(q)}A_T^{(q)}\beta_t^{(q)}$. Then $\tilde{f}'_{2m}(V_t^{(2m)}) = \tilde{f}''_{2m}(V_t^{(2m)}) = -G_t^{(q)}$. By Itô 's formula and

$$\begin{aligned} \tilde{f}_{2m}(s,V_s^{(2m)}) - \tilde{f}_{2m}(s,V_{s-}^{(2m)}) &= -(2m-\tilde{x})A_T^{(q)}\beta_s^{(q)}(M_s^{(q)} - M_{s-}^{(q)}) \\ &= -G_{s-}^{(q)}M_{s-}^{(q)}(e^{\Delta V_s^{(2m)}} - 1) \end{aligned}$$

for $\Delta M_s^{(q)} \neq 0$, we have

$$\begin{aligned} & p_t^{(2m),\tilde{x}} - p_0^{(2m),\tilde{x}} \\ &= \int_0^t -G_{s-}^{(q)}(q-1)\theta_q'\mathbf{S}_{s-}^{-1}d\langle M\rangle_s\lambda_s + \int_0^t -G_{s-}^{(q)}(q-1)\theta_q'\mathbf{S}_{s-}^{-1}dM_s, \end{aligned}$$

because from (15) and

$$e^{(q-1)g_q(x)}\mathrm{eg}_*(x) - 1 = (q-1)\theta_q'(e^x - \mathbf{1})\mu_*'(e^x-\mathbf{1}) + (q-1)\theta_q'(e^x-\mathbf{1}) + \mu_*'(e^x-\mathbf{1}) \tag{33}$$

we obtain

$$\mathrm{eg}_*(x) - \mathrm{eg}_*(x)e^{(q-1)g_q(x)} + e^{(q-1)g_q(x)} - 1 = -(q-1)\theta_q'(e^x-\mathbf{1})\mu_*'(e^x-\mathbf{1}).$$

It remains to show that $\vartheta^{(2m)} \in \mathcal{A}^{2m}$: We have with $dA_s = d\langle M_s\rangle\hat{\lambda}_s$

$$\int_0^T M_{s-}^{(q)}\hat{\lambda}_s' d\langle M\rangle_s\hat{\lambda}_s = \int_0^T (M_{s-}^{(q)})d\langle \int \hat{\lambda}_u' dM_u\rangle_s \le \sup_{s\le T} M_{s-}^{(q)}\langle \int \hat{\lambda}_u' dM_u\rangle_T.$$

The supremum of $M_t^{(q)}$ to the power p is integrable by Doob's inequality and the fact that $(q-1)p = q$. So $\|\vartheta^{(2m)}\|_{L^p(A)} < \infty$. Further,

$$[M]_t = \int_0^t \mathbf{S}_{u-}\sigma\sigma'\mathbf{S}_{u-}du + \int_0^t \int \mathbf{S}_{u-}(e^x-\mathbf{1})(e^x-\mathbf{1})'\mathbf{S}_{u-}N(dx,du).$$

It suffices to consider the jump part of $\int_0^T (\vartheta_t^{(2m)})' d[M]_t \vartheta_t^{(2m)}$.

$$
\begin{aligned}
&(\int_0^T (\vartheta_t^{(2m)})' d[M]_t \vartheta_t^{(2m)})^d \\
\leq\ & K_q \sup_{t\in[0,T]} |M_t^q|^2 \sup_{t\in[0,T]} |\beta_t^q|^2 \int_0^t \int \theta_q'(e^x - \mathbf{1})(e^x - \mathbf{1})'\theta_q N(dx, du) = (*)^2
\end{aligned}
$$

with a constant $K_q > 0$. So,

$$
\begin{aligned}
&E((*)^p) \\
\leq\ & K_{q,p} E(\sup_{t\in[0,T]} |M_t^q|^p| \int_0^t \int \theta_q'(e^x - \mathbf{1})(e^x - \mathbf{1})'\theta_q N(dx, du)|^{p/2}). \quad (34)
\end{aligned}
$$

In the one dimensional case, by Lemma 4.2 there exists an m_0 such that for all $m \geq m_0, \theta_{q_m} \to \theta_e$. If $\theta_e < 0$, then $\theta_{q_m} < 0$ for all $m \geq m_1 \geq m_0$. Jumps have to be bounded from above otherwise C_q fails. In view of the first assertion of the proof and because jumps are bounded from above, we have $\|\vartheta^{(2m)}\|_{L^p(M)} < \infty$. If $\theta_e > 0$, under assumption 4.2, all moments of $|\int_0^t \int \theta_q'(e^x - \mathbf{1})(e^x - \mathbf{1})'\theta_q N(dx, du)|$ exist and $E(\sup_{t\in[0,T]} |M_t^q|^{p(1+\epsilon)})$ is finite by assumption 4.2, see Lemma 4.3. The assertion follows by Hölder's inequality. In the multidimensional case, assumption 4.4 yields sufficient integrability of elements in (34). □

Next, we turn to the convergence of the solutions of the $2m$-level BSDEs to the BSDE of the exp-problem. We establish for all $r \geq 1$:

$$
E(\sup_t |p_t^{(2m),\tilde{x}} - p_t^{(\tilde{x})}|^r] \to 0, m \to \infty, \quad (35)
$$

where $p_t^{(\tilde{x})} := \tilde{x} + \int_0^t -\theta_q' \mathbf{S}_{s-}^{-1} dS$. As we already know the explicit form of $Z_{\min}$, it is easily seen that $p_T^{(\tilde{x})} = X_0^{(exp)}(\tilde{x})$. Hence the optimal portfolio that reaches $X_0^{(exp)}$ is equal to: $\vartheta^{(exp)} = -\theta_e' \mathbf{S}_{.-}^{-1} \in \mathcal{A}^{exp}$. Finally, we obtain $-\theta_e' \mathbf{S}_{.-}^{-1} \in \mathcal{A}^{exp}$ and

$$
p_T^{(2m),\tilde{x}} = X_0^{(2m)}(\tilde{x}) \xrightarrow{L^r} X_0^{(exp)}(\tilde{x}) \ r \geq 1.
$$

We get the following theorem:

Theorem 5.3. *Under the assumptions of Theorem 5.1,* $(-\theta_e' \mathbf{S}_{-t}^{-1}, 0) \in \mathcal{A}^{exp} \times \mathcal{K}$ *is the optimal portfolio of the problem*

$$
V_{exp}(\tilde{x}) = \max_{(\vartheta, C) \in \mathcal{A}^{exp} \times \mathcal{K}} E(1 - e^{-(\tilde{x} + \int_0^T \vartheta dS - C_T)}), \quad (36)
$$

where $\mathcal{K}$ *is an arbitrary class of right-continuous increasing processes. Further,*

$$
E(\sup_t |Y_t^{(2m),\tilde{x}} - (\tilde{x} + \int^t -\theta_e' \mathbf{S}_{s-}^{-1} dS)|^r] \to 0, \ m \to \infty, \ r \geq 1 \quad (37)
$$

where $Y_\cdot^{(2m),\tilde{x}}$ *is the optimal wealth process of*

$$V(\tilde{x})_{2m} \equiv \sup_{(\vartheta,C)\in\mathcal{A}^p\times\mathcal{K}^p} E[-(1-\frac{\tilde{x}+\int_0^T \vartheta dS - C_T}{2m})^{2m}],\ p=2m,\ \tilde{x}\leq 2m. \tag{38}$$

Finally, we establish the equality

$$X_0^{exp}(\tilde{x}) = \tilde{x} + \int_0^T -\theta_e' \mathbf{S}_{u-}^{-1} dS_u.$$

Proof. (Proof of Theorem 5.3) We proceed similar as in the proof of Lemma 4.4: By defining $g^{2m} := \frac{2m-\tilde{x}}{2m-1}\theta_q M^{(q)}\beta^{(q)}$, $q = 2m/(2m-1)$

$$\begin{aligned} & E(\sup_t |\int_0^t (\vartheta_s^{(2m)} - \theta_e' \mathbf{S}_{s-}^{-1}) dS_s|^r) \\ \leq\ & E(\sup_t |g_t^{2m} - \theta_e|^r \sup_{t\leq T} |\tilde{L}_t|^r) \leq K_r E \sup_t |g_t^{2m} - \theta_e|^{r(1+\epsilon)} \end{aligned}$$

by Hölder's inequality as in the proof of Lemma 5.2 and the following argument: Recall, $S = \mathcal{E}(\tilde{L})$. Split up $\tilde{L}$ in a martingale part $M^{(\tilde{L})}$ and a process of finite variation. The $E(\sup_{t\leq T}|M_t^{(\tilde{L})}|^r)$ is finite, by Doob's inequality and by Prop. 3.13 in [2] because $\int \|x\|^{\check{r}}(e^x-1)\nu(dx) < \infty$ with $\check{r} = r(1+\epsilon)$ by assumption 4.2/4.4. The part of finite variation is deterministic and independent of q, see (4). Hence $E(\sup|\tilde{L}_t|^{\check{r}})$ is finite. Further, we know that if a sequence of a stochastic process $(y_t^{(n)})_n$ converges in $S^{\check{r}}$ to y_t (supremums norm in $L^{\check{r}}$) and a deterministic sequence $(x_t^{(n)})_n$ converges in supremums norm to x_t, then the product converges in $S^{\check{r}}$ to $x_t y_t$. The deterministic part $\beta^{(q)}$ converges to 1 since it is continuous in t and $(q-1)g_q(x) \xrightarrow{q\downarrow 1} 0$. Fubini's Theorem can be applied as in Lemma 4.3 and 4.4, especially see (26). Finally, we need to consider the random part $M^{(q)}$. We show $E(\sup_t |M_t^{(q)} - 1|^{\check{r}}) \to 0$. By Doob's inequality we have

$$(E\sup_t |M_t^{(q)} - 1|)^{\check{r}} \leq E(\sup_t |M_t^{(q)} - 1|^{\check{r}}) \leq K_{\check{r}} E(|M_T^{(q)} - 1|^{\check{r}}).$$

So for every $\check{r} = r(1+\epsilon)$ there exists an $l \in \mathbb{N}$, $r(1+\epsilon) \leq 2l =: 2l(\check{r})$ such that

$$E(\sup_t |M_t^{(q)} - 1|^{\check{r}}) \leq K_l (E(|M_T^{(q)} - 1|^{2l}))^{\frac{\check{r}}{2l}}.$$

We need to prove that the last part converges to zero for every fixed $\check{r}$. We have

$$\begin{aligned} E(|M_T^{(q)} - 1|^{2l}) &= \sum_{k=0}^{2l} \binom{2l}{k} (-1)^k E((M_T^{(q)})^{2l-k}) \\ &\to \sum_{k=0}^{2l} \binom{2l}{k} (-1)^k (1)^{2l-k} = (-1+1)^{2l} = 0 \end{aligned}$$

if $E((M_T^{(q)})^u) \to 1$ for $u = 0, 1, ..., 2l$, which remains to show:

The Brownian and the jump part of $M^{(q)}$ are independent, by Yor's formula, we can split up $M_t^{(q)}$ in a continuous and discontinuous part

$$M_t^{(q)} = \mathcal{E}_t(-\int (q-1) - \theta_q' \sigma dW_s)\mathcal{E}_t(-\int\int (q-1)\theta_q'(e^x - \mathbf{1})\tilde{N}(dx, ds))$$

Both parts can be treated separately because of the independence. For the continuous part see [13] and observe that $\theta_q^{(i)} \in (\theta_e^{(i)} - \epsilon, \theta_e^{(i)} + \epsilon)$. The jump part is equal to $(M_t^{d,(q)})^u = e^{uX_t^{(q)}}$, where $X^{(q)}$ is a Lévy process with characteristic triplet (A_q, ν_q, b_q):

$$(0, \nu_q, -\int (e^{(q-1)g_q(x)} - 1 - (q-1)g_q(x)\mathbb{1}_{\|x\|\le 1})\nu(dx)),$$

where $\nu_q(A) = \mathbb{1}_{(q-1)g_q(x)\in A}$, $A \in B(\mathbb{R}^n)$. We have

$$\int_{\|x\|>1} e^{\check{r}(q-1)g_q(x)}\nu(dx) \le \nu(\{x : \|x\| \ge 1\}) + \int_{\|x\|>1} e^{qg_q(x)}\nu(dx) \quad (39)$$

for all $q \in (1, q(\check{r})]$, $q(\check{r})$ denotes the conjugate of $\check{r}$. The last integral is finite for all $q \in (1, q_0]$. Applying Prop. 3.14 in [2], we obtain:

$$E(e^{uX_t^{(q)}}) = e^{t\psi_q(-iu)}$$

where ψ_q denotes the characteristic function of $X^{(q)}$. By the Lévy- Khinchin representation, we get:

$$\begin{aligned}\psi_q(-iu) &= -u\int (e^{(q-1)g_q(x)} - 1 - (q-1)g_q(x)\mathbb{1}_{\|x\|\le 1})\nu(dx) \\ &\quad + \int (e^{u(q-1)g_q(x)} - 1 - u(q-1)g_q(x)\mathbb{1}_{\|x\|\le 1})\nu(dx)\end{aligned}$$

which is equal to zero for $u = 0, 1$ and converges to zero for $u = 2, ..., 2l$, because limit and integration can be changed as in (26). So for every $\check{r} = r(1+\epsilon)$ there exists an $l(r) \in \mathbb{N}$, $r(1+\epsilon) \le 2l(r)$ such that

$$E(\sup_t |M_t^{(q)} - 1|^{\check{r}})^{1/\check{r}} \le (E(|M_T^{(q)} - 1|^{2l}))^{\frac{1}{2l}} \to 0.$$

Apart from convergence in S^r of the price processes, we have uniform convergence of the integrands. This further yields uniform convergence in probability of the integrals and therefore of the price processes. Finally, $-\theta_e{}'\mathbf{S}_-^{-1} \in \mathcal{A}^{exp}$, as in the proof of Lemma 5.2. □

We finally apply the results of this paper to the Brownian-Poisson case:

Example 5.1. *(Brownian Motion + Compound Poisson) Let γ be the intensity and f the jump size distribution of a compound Poisson process. $\sigma \neq 0$ the volatility of a Brownian motion W and β defined as before*

$\beta = -(\mu - r) = -(b - \frac{1}{2}\sigma^2 - r)$. *All assumptions of Theorem 5.1 are satisfied. We get* $\vartheta_t^{(exp)} = -S_{t-}^{-1}\theta_e$ *where* θ_e *is the solution of (12):*

$$\theta_e\sigma^2 + \int((e^x - 1)e^{\theta_e(e^x-1)} - 1_{|x|\leq 1}x)\gamma f(dx) = \beta$$

We present an example when also the variance optimal measure is equivalent to P*. In the case of a truncated normal distribution (jump size is bounded by an* $L \in \mathbb{R}$*), we have to ensure that* $\mu_* \in ((1-e^L)^{-1}, 1),\ L > 0$ *or* $\mu_* \in (-\infty, 1),\ L < 0$*. The upper bound is always satisfied.* μ_L, σ_L^2 *and* γ *have to be chosen appropriately such that* $\mu_* \in ((1-e^L)^{-1}, 1)$*, where*

$$\mu_* \quad = \quad \frac{\beta - \frac{\gamma}{N_{\mu_L,\sigma_L}(L)}(\mu_{LN,1} - 1 - \int_{-1}^{1} x n_{\mu_L,\sigma_L}(x)dx)}{(\sigma^2 + \gamma\mu_{LN,2} - 2\gamma\mu_{LN} + \gamma)}$$

and $\mu_{LN,n}$ *denotes the nth moment of* $x = e^y$ *with* $y \sim N(\mu_L, \sigma_L^2)$*. Choose e.g.* $L = 0.75,\ \gamma = 0.001,\ \ \mu = b + \frac{1}{2}\sigma^2 = 0.09,\ \sigma^2 = 0.101,\ \sigma_L^2 = 0.07,$ *or* $\sigma = 0.3178,\ \sigma_L = 0.2646$*. A slight increase of some parameters and the condition fails! The coefficients above imply* $\vartheta^{(exp)}S = -\theta_e = 0.8923$ *compared to the continuous portfolio* $\vartheta_c^{(exp)}S_c = \mu/\sigma^2 = 0.8911$*. Hence, in this case we invest more in the stock if jumps are considered, because the jump measure is defined on* x *and not on* $y = e^x - 1$*.*

Alternatively, for a usual Poisson process, we just have one jump size, say a*. For* $a < 0$*, e.g. to simulate a sudden jump to default, the variance optimal measure is always equivalent to* P*. For* $a > 0$*, we need to ensure that*

$$a \in (0,1)(\mu - r) + \sigma^2/(1-e^a) < a\gamma,\ a > 1 : (\mu - r)(e^a - 1) \leq \sigma^2.$$

A very low constant a, say -5, can be interpreted as a default. So when γ *increases a lot, the portfolio becomes negative, see figure 5.1.*

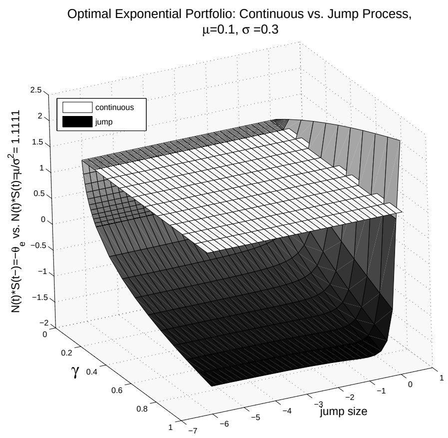

Figure 1. Influence of jump size and intensity upon the optimal portfolio

■

6. Conclusion

Kohlmann and Niethammer [13] provide a framework to solve the dynamic utility maximization problem for an exponential utility function via an approximation approach. Optimal terminal values as well as its optimal portfolios are considered. However, this is done in a continuous semimartingale setting. We are able to expand this result to exponential Lévy processes. Derivations of the convergence of the static problems, i.e. the optimal terminal wealth, as well as the portfolios and its convergence are given. Results from the continuous setting on the sequence $-(1-\frac{x}{2m})^{2m}$ (2m-th problem) can be almost overtaken to solve the static problem, whereas proofs of our convergence results become quite different. To obtain convergence several additional results on q-optimal and the minimal entropy martingale measure are proven. Finally, our convergence result yields a construction method for an explicit portfolio in the exponential case with jumps.

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